GIS

UNIVERSITÉ DU LUXEMBOURG

Geodätisches Institut – Universität Stuttgart

Converted Total Least Squares method and Gauss-Helmert model with applications to 3-D coordinate transformations

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Session: Positioning, Navigation & Reference Systems

Second DAAD TN Workshop, Luxemburg, July 24–28, 2018

Contents

1. Review of Total Least Squares methods

■ History of total least squares

- In **computational mathematics and engineering**: **Total Least Squares (TLS)** - method of fitting that is appropriate when there are errors in both the observation vector and in the design matrix;
- In the **statistical community**: **Errors-In-Variables (EIV)** modelling or **orthogonal regression;**
- The TLS/EIV principle was studied by *Aocock* (1878) and *Pearson* (1902), already **more than one century ago**;
- Widely used only **since 1980s** – one of the main reason: the availability of efficient and numerical robust algorithms, e.g. **SVD;**
- In geodetic data analysis, TLS has been intensively studied and several approaches have also been developed **since 2000**
- The proper application of TLS method in geodesy: **coordinate transformation,** in which the old local coordinates with lower accuracy need to be transformed to a higher precision newer network.

The estimators of LS and TLS methods-1

- **Least Squares estimator (LS):**

$$\mathbf{y} - \mathbf{e} = \mathbf{A}\xi$$

$$E\{\mathbf{e}\}=0,\ \ D\{\mathbf{e}\}=\mathbf{\Sigma}_{\mathbf{y}}$$

- **Total Least Squares estimator (TLS):**

$$(\mathbf{y} - \mathbf{e}) = (\mathbf{A} - \mathbf{E}_{\mathbf{A}})\,\xi$$

$$E\{[(\mathrm{vec}\mathbf{E}_{\mathbf{A}}), \mathbf{e}]\}=0,\ \ C\{\mathrm{vec}\mathbf{E}_{\mathbf{A}}, \mathbf{e}\}=0,$$

$$D\{\mathrm{e}\}=\mathbf{\Sigma}_0 \otimes \mathbf{Q}_{\mathbf{y}},\ \ D\{\mathrm{vec}\mathbf{E}_{\mathbf{A}}\}=\mathbf{\Sigma}_0 \otimes \mathbf{Q}_{\mathbf{a}}$$

The estimators of LS and TLS methods-2

- **Least Squares estimator (LS) and the solution:**

$$\boxed{\mathbf{e}^T\mathbf{e} = \min(\mathbf{e}, \xi)}$$

$$\xi_{LS} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

- **Total Least Squares estimator (TLS) and the solution**

The Total Least Squares Euler-Lagrange Approach
for $\mathbf{Q}_{\mathbf{y}} = \mathbf{I}$ and $\mathbf{Q}_{\mathbf{E}_{\mathbf{A}}} = \mathbf{I}$

$$\boxed{\mathbf{e}^T\mathbf{e} + (\mathrm{vec}\,\mathbf{E}_{\mathbf{A}})^T(\mathrm{vec}\,\mathbf{E}_{\mathbf{A}}) = \min(\mathbf{e}, \mathbf{E}_{\mathbf{A}}, \xi).}$$

- One solution of the TLS problem is by making substantial use of the singular value decomposition (SVD) (van Huffel and Zha, 1993);

- Schaffrin (2005) introduced a solution of the TLS problem by iteration procedures.

Closed-form solution of TLS

Closed-form expression of the basic TLS solution by making substantial use of the singular value decomposition (SVD) (van Huffel and Zha, 1993):

$$\boxed{\boldsymbol{\xi}_{TLS} = (\mathbf{A}^T\mathbf{A} - \sigma_{n+1}{}^2\mathbf{I})^{-1}\mathbf{A}^T\mathbf{y}}$$

with σ_{n+1} the smallest singular value of the augmented data matrix $[\mathbf{A};\mathbf{y}]$:

$$[\mathbf{A};\mathbf{y}] = \mathbf{U\Sigma V}^T = \sum_{i=0}^{n+1}\sigma_i u_i v_i^T,\ \ \sigma_1 \geq \cdots \geq \sigma_{n+i} \geq 0.$$

The best TLS approximation $[\hat{\mathbf{A}};\hat{\mathbf{y}}]$ of $[\mathbf{A};\mathbf{y}]$ is given by

$$[\hat{\mathbf{A}};\hat{\mathbf{y}}] = \mathbf{U}\hat{\mathbf{\Sigma}}\mathbf{V}^T,\ \ \text{with}\ \ \hat{\mathbf{\Sigma}} = diag(\sigma_1,\cdots,\sigma_n,0)$$

and with corresponding TLS correction matrix

$$[\hat{\mathbf{E}}_{\mathbf{A}};\hat{\mathbf{e}}] = [\mathbf{A};\mathbf{y}] - [\hat{\mathbf{A}};\hat{\mathbf{y}}] = \sigma_{n+1}u_{n+1}v_{n+1}^T.$$

- ***Easily implemented with MATLAB program in the calculation and bring meaningful application possibility in data analysis!***
- ***Also easily applied for the case that observations with variance-covariance!***

A solution by iteration (Schaffrin, 2005)

- **1. Compute the LS solution:** $\hat{\boldsymbol{\xi}}^{1} = (\mathbf{A}^{T}\mathbf{A})^{-1} \times \mathbf{A}^{T}\mathbf{y}$

- **2. Insert the solution of step 1. as the initial value for the following iterative process**

$$\boxed{\hat{\boldsymbol{\xi}}^{i+1} = (\mathbf{A}^{T}\mathbf{A})^{-1} \times \left[\mathbf{A}^{T}\mathbf{y} + \hat{\boldsymbol{\xi}}^{i} \times \frac{(\mathbf{y} - \mathbf{A} \times \hat{\boldsymbol{\xi}}^{i})^{T} \times (\mathbf{y} - \mathbf{A} \times \hat{\boldsymbol{\xi}}^{i})}{(\mathbf{1} + (\hat{\boldsymbol{\xi}}^{i})^{T} \times \hat{\boldsymbol{\xi}})} \right]}$$

- **3. End when :** $\left\| \hat{\boldsymbol{\xi}}^{i+1} - \hat{\boldsymbol{\xi}}^{i} \right\| < \varepsilon$

■ **Partial-EIV model** (Xu, Liu and Shi, 2012, Wang, Li, Liu, 2015)

- Reform the EIV model $\mathbf{y}-\mathbf{e}_{\mathbf{y}}=\left(\mathbf{A}-\mathbf{E}_{\mathbf{A}}\right)\xi$ into a partial-EIV model by extracting functionally independent random variables within the design matrix:

$$\boxed{\mathbf{y}-\mathbf{e}_{\mathbf{y}}=(\xi^{T}\otimes\mathbf{I}_{m})[\mathbf{h}+\mathbf{B}(\mathbf{a}-\mathbf{e}_{\mathbf{a}})]}$$

- The iterative process will implemented with the following steps

1) The initial values of parameters ξ_1 can be taken from LS solution

$$\xi_1=(\mathbf{A}^{T}\mathbf{P}_{\mathbf{y}}\mathbf{A})^{-1}\mathbf{A}^{T}\mathbf{P}_{\mathbf{y}}\mathbf{y}$$

2) Get the correspond cofactor matrix of $\mathbf{y}$

$$\mathbf{Q}_{c(i)}=\mathbf{Q}_{\mathbf{y}}+(\xi_{(i)}^{T}\otimes\mathbf{I}_{m})\mathbf{B}\mathbf{Q}_{a}\mathbf{B}^{T}(\xi_{(i)}\otimes\mathbf{I}_{m})$$

3) Calculate the value differences $\delta\hat{\xi}$ and get the new value $\hat{\xi}$

$$\delta\hat{\xi}_{(i+1)}=(\mathbf{A}_{(i)}^{T}\mathbf{Q}_{c(i)}^{-1}\mathbf{A}_{(i)})^{-1}\mathbf{A}_{(i)}^{T}\mathbf{Q}_{c(i)}^{-1}(\mathbf{y}-\mathbf{A}\xi_{(i)})$$
$$\hat{\xi}_{(i+1)}=\delta\hat{\xi}_{(i+1)}+\xi_{(i)}$$

4) Calculate the correction of $\mathbf{y}$ and $\mathbf{a}$

$$\hat{\mathbf{e}}_{\mathbf{y}(i+1)}=\mathbf{Q}_{\mathbf{y}}\mathbf{Q}_{c(i)}^{-1}(\mathbf{y}-\mathbf{A}\xi_{(i)}-\mathbf{A}_{(i)}\delta\hat{\xi}_{(i+1)})$$
$$\hat{\mathbf{e}}_{\mathbf{a}(i+1)}=\mathbf{Q}_{\mathbf{a}(i+1)}\mathbf{B}^{T}(\xi_{(i)}\otimes\mathbf{I}_{m})\mathbf{Q}_{c(i)}^{-1}(\mathbf{y}-\mathbf{A}\xi_{(i)}-\mathbf{A}_{(i)}\delta\hat{\xi}_{(i+1)})$$

5) Repeat steps 2)-4), until $\left\|\delta\hat{\xi}_{(i+1)}\right\|<\varepsilon$ for a given ε.

■ Problems with TLS solutions

- **SVD:**
  - non-random elements wrongly corrected
  - repetition of random elements
  - different corrections for the same element
- **Iterative approach:**
  - repetition of random elements
  - sometimes the iteration may not be convergent
- **Partial-EIV:**
  - sometimes the iteration may not be convergent

2. Converted Total Least Squares method (CTLS) and implementation in coordinate transformation

- **Firstly take Gauss-Makov model as basic observation equation:**

$$\mathbf{y} = \mathbf{A}\boldsymbol{\xi} + \mathbf{e_y} \qquad (1)$$

- **Augmenting the observation equations with taking random design matrix elements as virtual observations:**

$$\mathbf{y_a} = \boldsymbol{\xi}_\mathbf{a} + \mathbf{e_a} \qquad (2)$$

- **Combinate the two equations together:**

$$\boxed{\begin{cases} \mathbf{y} = \mathbf{A}\boldsymbol{\xi} + \mathbf{e_y} \\ \mathbf{y_a} = \boldsymbol{\xi}_\mathbf{a} + \mathbf{e_a} \end{cases}} \qquad (3)$$

Here, the matrix A is used to denote the design matrix, which is formed by the initial value of elements in $\boldsymbol{\xi}_\mathbf{a}$.

■ Converted Total Least Squares method (CTLS)-2

- **From the above and according to the joint research results (Yao, Cai, Kong and Sneeuw, 2010) we can get the following derivations**

$$
\begin{aligned}
\mathbf{e}_{\mathbf{y}} &= \left(\mathbf{A}_{\xi}^{0} + \mathbf{E}_{\mathbf{A}}\right)\left(\xi^{0} + \Delta\xi\right) - \mathbf{y} \\
&= \mathbf{A}_{\xi}^{0}\Delta\xi + \mathbf{E}_{\mathbf{A}}\xi^{0} + \mathbf{A}_{\xi}^{0}\xi^{0} - \mathbf{y} + \mathbf{E}_{\mathbf{A}}\Delta\xi \qquad \rightarrow \mathbf{E}_{\mathbf{A}}\Delta\xi \approx 0 \\
&= \mathbf{A}_{\xi}^{0}\Delta\xi + \mathbf{B}\Delta\mathbf{a} + \mathbf{A}_{\xi}^{0}\xi^{0} - \mathbf{y} \qquad (4) \\
\mathbf{e}_{\mathbf{a}} &= \xi_{\mathbf{a}} - \mathbf{y}_{\mathbf{a}} \\
&= (\mathbf{a}^{0} + \Delta\mathbf{a}) - \mathbf{y}_{\mathbf{a}} \\
&= \Delta\mathbf{a} + (\mathbf{a}^{0} - \mathbf{y}_{\mathbf{a}})
\end{aligned}
$$

With the vectorization of the matrix product equation

$$
\underset{n\cdot t}{\mathbf{B}}\,\underset{t\cdot 1}{\mathbf{x}} = \underset{n\otimes np}{(\mathbf{x}^{T} \otimes \mathbf{I})}\,\underset{np\cdot 1}{\mathrm{vec}(\mathbf{B})}
$$

- where $\mathbf{E}_{\mathbf{A}}$ is composed of $\Delta\mathbf{a}$, the corrections to the new paramaters, and $\mathbf{B}\Delta\mathbf{a}$ is rewritten form of $\mathbf{E}_{\mathbf{A}}\xi^{0}$, which is the key step of this approach;
- $\mathbf{A}_{\xi}^{0}$ is composed of non-stochastic elements in the design matrix and the initial values in $\mathbf{a}$.

■ Implementation of CTLS method to 3-D 7-Parameter Helmert transformation

Three-dimensional model: 7-Parameter Helmert Transformation

(3 translations, 3 rotations, 1 scale correction) also called *Bursa-Wolf model*)

$$\boxed{\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = (1+dm)\begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}\begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}}$$

➢ **Centering the coordinates at the midpoint:**

$$\underbrace{\begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix}_i}_{\mathbf{y}_i} =: \underbrace{\begin{bmatrix} 0 & -z_L & y_L & x_L \\ z_L & 0 & -x_L & y_L \\ -y_L & x_L & 0 & z_L \end{bmatrix}_i}_{\mathbf{A}_i} \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ dm \end{bmatrix}}_{\xi} \quad \text{with} \quad \begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix}_i = \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix}_i - \begin{bmatrix} \bar{X}_G \\ \bar{Y}_G \\ \bar{Z}_G \end{bmatrix}, \quad \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix}_i = \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix}_i - \begin{bmatrix} \bar{X}_L \\ \bar{Y}_L \\ \bar{Z}_L \end{bmatrix}$$

➢ **For empirical coordinate transformations:**

$$E\left\{\begin{bmatrix} x_1 \\ \dots \\ x_n \\ y_1 \\ \dots \\ y_n \\ z_1 \\ \dots \\ z_n \end{bmatrix}_G\right\} =: E\left\{\begin{bmatrix} 0 & -z_1 & y_1 & x_1 \\ \dots & \dots & \dots & \dots \\ 0 & -z_n & y_n & x_n \\ z_1 & 0 & -x_1 & y_1 \\ \dots & \dots & \dots & \dots \\ z_n & 0 & -x_n & y_n \\ -y_1 & x_1 & 0 & z_1 \\ \dots & \dots & \dots & \dots \\ -y_n & x_n & 0 & z_n \end{bmatrix}_L\right\}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ dm \end{bmatrix}.$$

In Matrix Notation:

$$\boxed{(\mathbf{y}\text{-}\mathbf{e}) = (\mathbf{A}-\mathbf{E}_\mathbf{A})\,\xi}$$

$$E\{[(\mathrm{vec}\mathbf{E}_\mathbf{A}),\mathbf{e}]\}=0, \quad C\{\mathrm{vec}\mathbf{E}_\mathbf{A},\mathbf{e}\}=0,$$

$$D\{\mathrm{e}\}=\mathbf{\Sigma}_0\otimes\mathbf{Q}_\mathbf{y}, \quad D\{\mathrm{vec}\mathbf{E}_\mathbf{A}\}=\mathbf{\Sigma}_0\otimes\mathbf{Q}_\mathbf{a}$$

■ Implementation of CTLS method:

➢ **Example for converting $\mathbf{E_A}\boldsymbol{\xi}^0$ to $\mathbf{B}\Delta\mathbf{a}$ within 3D Helmert transformation**

$$\mathbf{E_A}=\begin{bmatrix} 0 & -\Delta z_{l1} & \Delta y_{l1} & \Delta x_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -\Delta z_{ln} & \Delta y_{ln} & \Delta x_{ln} \\ \Delta z_{l1} & 0 & -\Delta x_{l1} & \Delta y_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta z_{ln} & 0 & -\Delta x_{ln} & \Delta y_{ln} \\ -\Delta y_{l1} & \Delta x_{l1} & 0 & \Delta z_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ -\Delta y_{ln} & \Delta x_{ln} & 0 & \Delta z_{ln} \end{bmatrix} \qquad \boldsymbol{\xi}^0=\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ dm \end{bmatrix}=\begin{bmatrix} \xi_1^0 \\ \xi_2^0 \\ \xi_3^0 \\ \xi_4^0 \end{bmatrix} \qquad \Delta\mathbf{a}=\begin{bmatrix} \Delta x_{l1} \\ \vdots \\ \Delta x_{ln} \\ \Delta y_{l1} \\ \vdots \\ \Delta y_{ln} \\ \Delta z_{l1} \\ \vdots \\ \Delta z_{ln} \end{bmatrix}$$

$$\mathbf{E_A}\boldsymbol{\xi}^0=\mathbf{B}\Delta\mathbf{a}=\left(\begin{bmatrix} \xi_4^0 & \xi_2^0 & -\xi_3^0 \\ -\xi_2^0 & \xi_4^0 & \xi_1^0 \\ \xi_3^0 & -\xi_1^0 & \xi_4^0 \end{bmatrix}\otimes\mathbf{I}_n\right)\Delta\mathbf{a}$$

$$\underset{3n\times 3n}{\mathbf{B}}=\begin{bmatrix} \xi_4^0 & \xi_2^0 & -\xi_3^0 \\ -\xi_2^0 & \xi_4^0 & \xi_1^0 \\ \xi_3^0 & -\xi_1^0 & \xi_4^0 \end{bmatrix}\otimes\mathbf{I}_n=\begin{bmatrix} \xi_4^0 & 0 & 0 & \xi_2^0 & 0 & 0 & -\xi_3^0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \xi_4^0 & 0 & 0 & \xi_2^0 & 0 & 0 & -\xi_3^0 \\ -\xi_2^0 & 0 & 0 & \xi_4^0 & 0 & 0 & \xi_1^0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & -\xi_2^0 & 0 & 0 & \xi_4^0 & 0 & 0 & \xi_1^0 \\ \xi_3^0 & 0 & 0 & -\xi_1^0 & 0 & 0 & \xi_4^0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \xi_3^0 & 0 & 0 & -\xi_1^0 & 0 & 0 & \xi_4^0 \end{bmatrix}$$

■ Converted Total Least Squares method (CTLS)-3

- Define

$$\mathbf{z}=\begin{bmatrix}\mathbf{y}-\mathbf{A}_{\xi}^{0}\boldsymbol{\xi}^{0}\\ \mathbf{y}_{a}-\mathbf{a}\end{bmatrix},\ \mathbf{A}_{\eta}=\begin{bmatrix}\mathbf{A}_{\xi}^{0} & \mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{bmatrix},\ \Delta\boldsymbol{\eta}=\begin{bmatrix}\Delta\boldsymbol{\xi}\\ \Delta\mathbf{a}\end{bmatrix},\ \mathbf{e}_{z}=\begin{bmatrix}\mathbf{e}_{y}\\ \mathbf{e}_{a}\end{bmatrix}$$

- which can be presented as

$$\boxed{\mathbf{z}=\mathbf{A}_{\eta}\Delta\boldsymbol{\eta}+\mathbf{e}_{z}}$$

with the new weight matrix

$$\mathbf{P}_{z}=\begin{bmatrix}\mathbf{P}_{y} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{a}\end{bmatrix},\ \text{or}\ \boldsymbol{\Sigma}_{z}=\sigma_{z_0}^{2}\begin{bmatrix}\mathbf{P}_{y}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{a}^{-1}\end{bmatrix}$$

- The estimation criterion is still to get the minimum of the residual sums of the squares

$$\boxed{\mathbf{e}_{z}^{T}\mathbf{P}_{z}\mathbf{e}_{z}=\mathbf{e}_{y}^{T}\mathbf{P}_{y}\mathbf{e}_{y}+\mathbf{e}_{a}^{T}\mathbf{P}_{a}\mathbf{e}_{a}\rightarrow\min}$$

- The TLS problem can be estimated considering the weight of observations and stochastic design matrix by:

$$\boxed{\Delta\hat{\boldsymbol{\eta}}=\left(\mathbf{A}_{\eta}^{T}\mathbf{P}_{z}\mathbf{A}_{\eta}\right)^{-1}\mathbf{A}_{\eta}^{T}\mathbf{P}_{z}\mathbf{z}}$$

3. Comparison and analysis of the results with CTLS and other TLS methods

- **Statistical data by the quadratics sums of the residuals for 4 estimator:**
  ($\hat{\mathbf{e}}$ - the residuals of observation and $\hat{\mathbf{E}}, \hat{\mathbf{e}}_{\mathbf{a}}$ - the residuals of design matrix)

LS:

$$\hat{\mathbf{e}}_{LS}^T \hat{\mathbf{e}}_{LS} = 4.063234 (m^2)$$

TLS(SVD):

$$\hat{\mathbf{e}}_{TLS}^T \hat{\mathbf{e}}_{TLS} = 1.015790 (m^2)$$

$$\hat{\mathbf{E}}_{TLS}^T \hat{\mathbf{E}}_{TLS} = 1.015808 (m^2)$$

$$\hat{\mathbf{e}}_{TLS}^T \hat{\mathbf{e}}_{TLS} + \hat{\mathbf{E}}_{TLS}^T \hat{\mathbf{E}}_{TLS} = 2.031598 (m^2)$$

Partial-EIV:

$$\hat{\mathbf{e}}_{TLSP}^T \hat{\mathbf{e}}_{TLSP} = 1.015790 (m^2)$$

$$\hat{\mathbf{e}}_{\mathbf{a}_{TLSP}}^T \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}} = 1.015808 (m^2)$$

$$\hat{\mathbf{e}}_{TLSP}^T \hat{\mathbf{e}}_{TLSP} + \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}}^T \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}} = 2.031598 (m^2)$$

CTLS:

$$\hat{\mathbf{e}}_{CTLS}^T \hat{\mathbf{e}}_{CTLSP} = 1.015790 (m^2)$$

$$\hat{\mathbf{e}}_{\mathbf{a}_{CTLS}}^T \hat{\mathbf{e}}_{\mathbf{a}_{CTLS}} = 1.015808 (m^2)$$

$$\hat{\mathbf{e}}_{TLSP}^T \hat{\mathbf{e}}_{TLSP} + \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}}^T \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}} = 2.031598 (m^2)$$

■ Statistical comparison the results of the coordinate transformation with different estimation methods

Transformation models	Collocated sites	Absolute mean Residuals(m) $[V_N]$ $[V_E]$	Max. of absolute Residuals(m) $[V_N]$ $[V_E]$	RMS (m)	Standard deviation of unit weight(m)
LS	B-W 131	0.1051 0.0843	0.4212 0.3112	0.1240	0.1026
TLS	B-W 131	0.0526 0.0843	0.2106 0.1556	0.0620	0.0513
Partial-EIV	B-W 131	0.0526 0.0843	0.2106 0.1556	0.0620	0.0513
CTLS	B-W 131	0.0526 0.0843	0.2106 0.1556	0.0620	0.0513

131 BWREF Points	7-parameter Helmert transformation GK (DHDN)-UTM (ETRS89)						
	$T_x(m)$	$T_y(m)$	$T_z(m)$	$\alpha('')$	$\beta('')$	$\gamma('')$	$dm(\times 10^{-6})$
LS	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208
TLS	582.901702	112.168078	405.603061	-2.255032	-0.335003	2.068369	9.117210
Partial-EIV	582.901701	112.168078	405.603061	-2.255032	-0.335003	2.068369	9.117210
CTLS	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208

Comparison the results of the coordinate transformation between LS and classic TLS

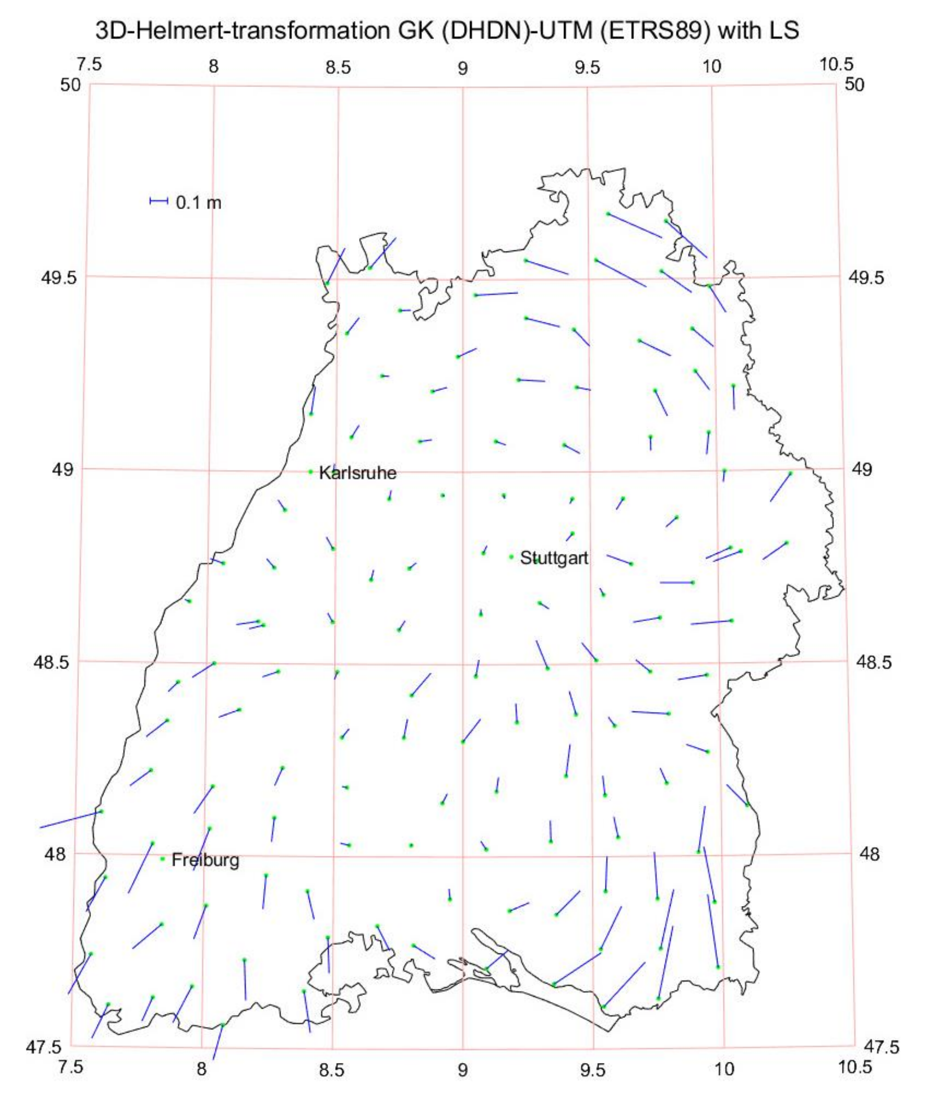

Fig. 1 LS

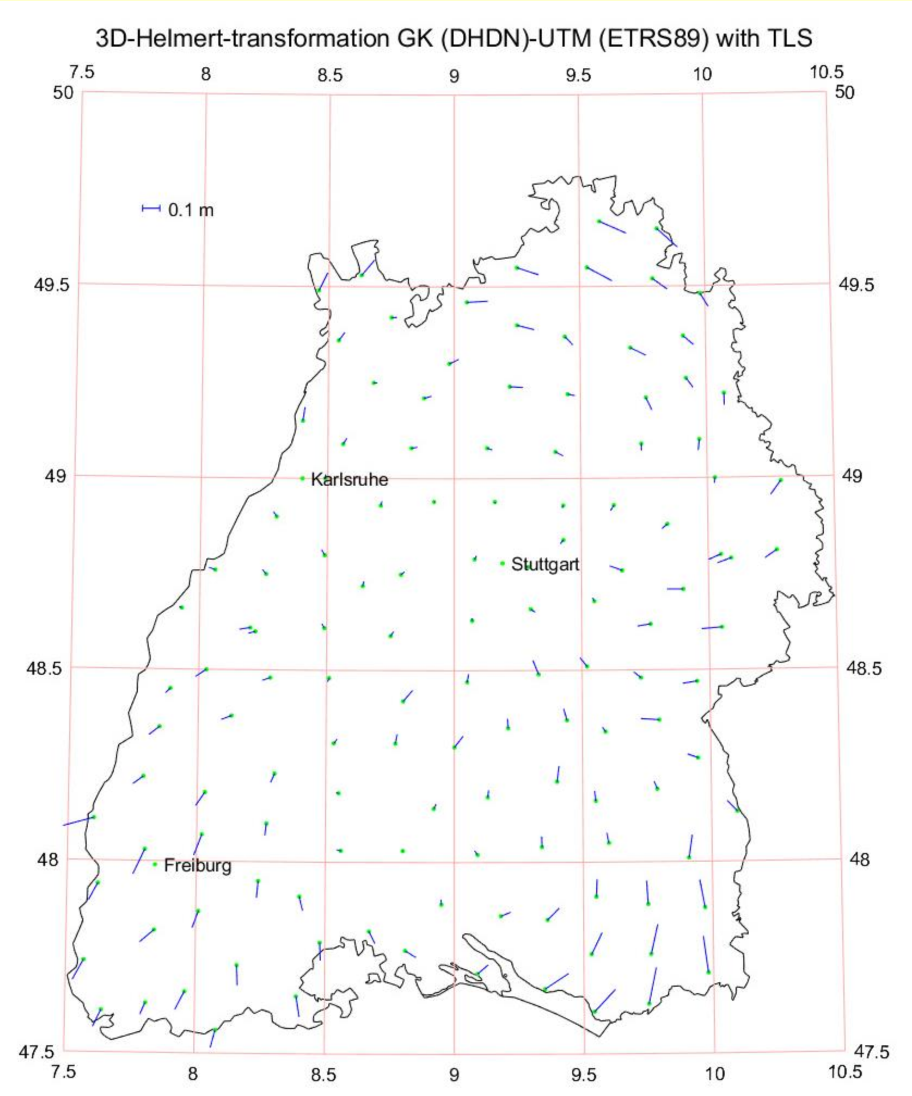

Fig. 2 TLS

■ Comparison the results of coordinate transformation between partial TLS and converted TLS

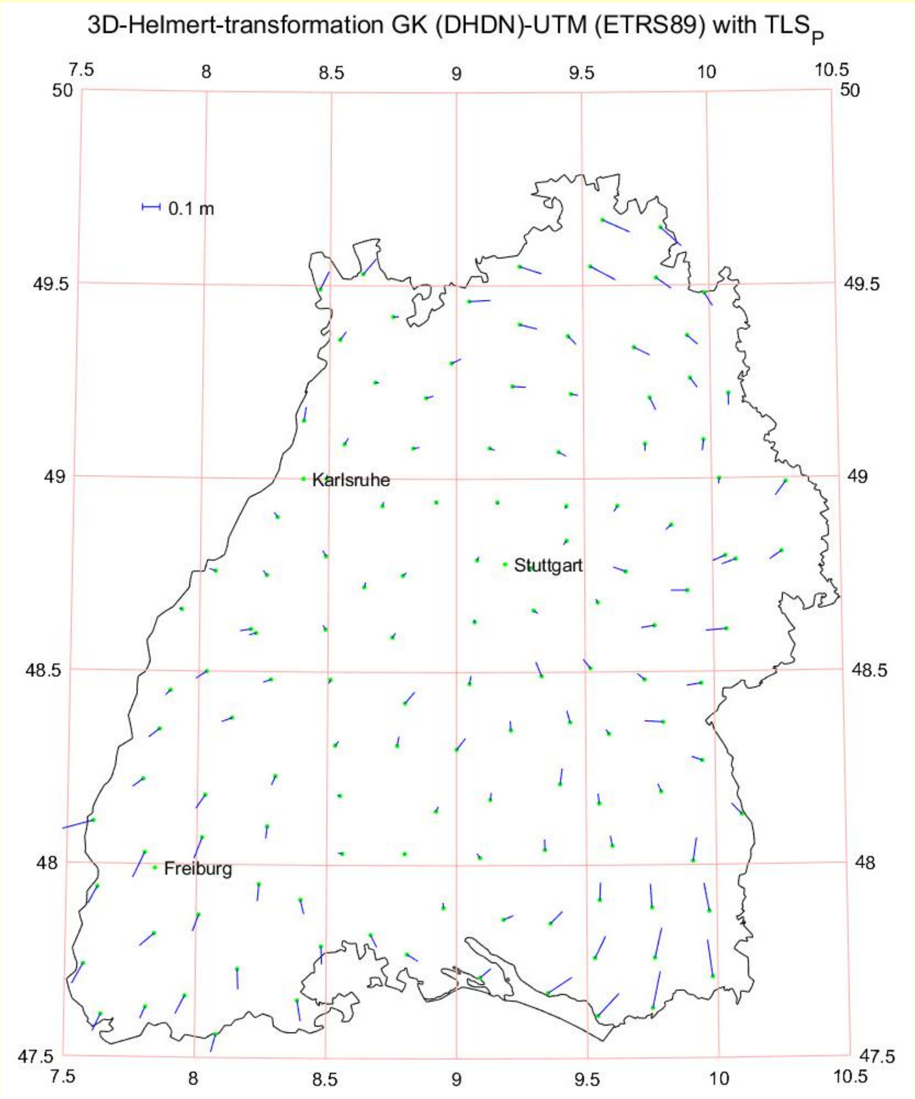

Fig. 3 TLSP

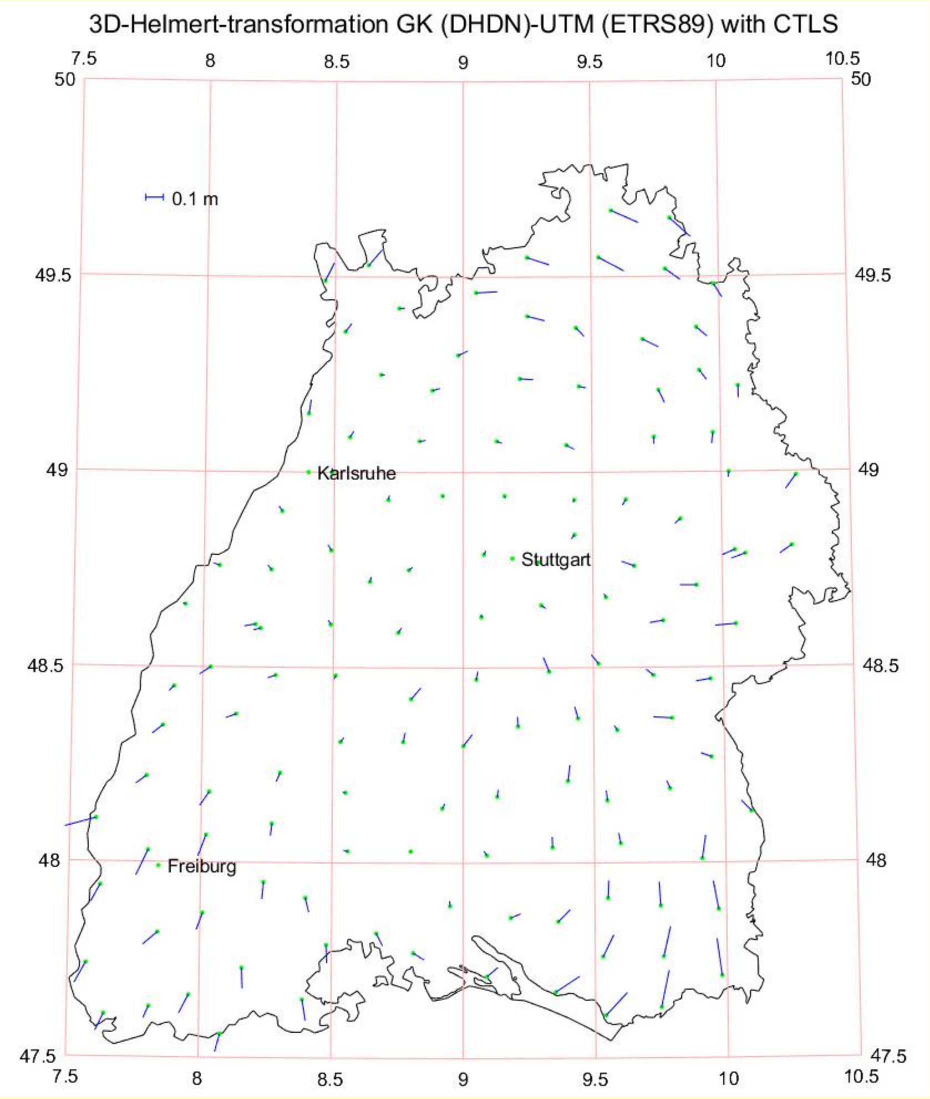

Fig. 4 CTLS

4. Connection of CTLS estimator and the estimator of Gauss-Helmert models

Thr … ation

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Räumliche Helmert-Transformation variabler Koordinaten im Gauß-Helmert- und im Gauß-Markoff-Modell

Karl-Rudolf Koch

127. Jg. 3/2002 zfv 147

- **Ko … del for the case of coordinate transformation directly**

$$\mathbf{y} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\boldsymbol{\gamma} + \mathbf{e}_{\mathbf{y}}$$
$$\text{with } E\{\boldsymbol{\gamma}\} = \mathbf{0} \text{ and } D\{\boldsymbol{\gamma}\} = \sigma^2 \boldsymbol{\Sigma}_{\gamma\gamma} \qquad (1)$$

- **He converted Gauss-Helmert model into Gauss-Markov model for the same case of transformation with additional new parameters/observations**

$$\boxed{\begin{cases} \mathbf{y} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\boldsymbol{\gamma} + \mathbf{e}_{\mathbf{y}} \\ \mathbf{y}_{\gamma} = \boldsymbol{\gamma} + \mathbf{e}_{\gamma} \end{cases}} \qquad (2)$$

- **He has also proved that both the G-H-M and converted G-M-M produce identical estimated parameters!**

■ Connection of CTLS estimator and the estimator of Gauss-Helmert models

- According to our new development of CTLS, in which we startend in deailing with the reformed empirical transformation

$$E\left\{\begin{bmatrix} x_1 \\ \dots \\ x_n \\ y_1 \\ \dots \\ y_n \\ z_1 \\ \dots \\ z_n \end{bmatrix}_G\right\} =: E\left\{\begin{bmatrix} 0 & -z_1 & y_1 & x_1 \\ \dots & \dots & \dots & \dots \\ 0 & -z_n & y_n & x_n \\ z_1 & 0 & -x_1 & y_1 \\ \dots & \dots & \dots & \dots \\ z_n & 0 & -x_n & y_n \\ -y_1 & x_1 & 0 & z_1 \\ \dots & \dots & \dots & \dots \\ -y_n & x_n & 0 & z_n \end{bmatrix}_L\right\}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ dm \end{bmatrix}. \quad \Rightarrow \mathbf{e_y} = \left(\mathbf{A}_{\xi}^{0} + \mathbf{E_A}\right)\left(\boldsymbol{\xi}^{0} + \Delta\boldsymbol{\xi}\right) - \mathbf{y}$$

- After the convert of the random elements in design matrix into new random parameters together with the argumention of virtual observations to Gauss-Markov model we arrives similar models:

$$\begin{cases} -\mathbf{e_y} = \mathbf{A}_{\xi}^{0}\Delta\boldsymbol{\xi} + \mathbf{E_A}\boldsymbol{\xi}^{0} + \mathbf{A}_{\xi}^{0}\boldsymbol{\xi}^{0} - \mathbf{y} + \mathbf{E_A}\Delta\boldsymbol{\xi}^{0} \\ \qquad = \mathbf{A}_{\xi}^{0}\Delta\boldsymbol{\xi} + \mathbf{B}\Delta\mathbf{a} + \mathbf{A}_{\xi}^{0}\boldsymbol{\xi}^{0} - \mathbf{y} \\ -\mathbf{e_a} = \boldsymbol{\xi}_{\mathbf{a}} - \mathbf{y_a} \\ \qquad = \Delta\mathbf{a} + (\mathbf{a}^{0} - \mathbf{y_a}) \end{cases}$$

CTLS

$$\begin{cases} \mathbf{y} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\boldsymbol{\gamma} + \mathbf{e_y} \\ \mathbf{y}_{\gamma} = \qquad \boldsymbol{\gamma} + \mathbf{e}_{\gamma} \end{cases}$$

Gauss-Helmert model

■ Statistical comparison of the estimated results of CTLS and the Gauss-Helmert models

Transformation models	Collocated sites	Absolute mean Residuals(m) $[V_N]$ $[V_E]$	Max. of absolute Residuals(m) $[V_N]$ $[V_E]$	RMS (m)	Standard deviation of unit weight(m)
CTLS	B-W 131	0.0526 0.0843	0.2106 0,1556	0.0620	0.0513
Gauss-Helmert	B-W 131	0.0526 0.0843	0.2106 0,1556	0.0620	0.0513

131 BWREF Points	7-parameter Helmert transformation GK (DHDN)-UTM (ETRS89)						
	$T_x(m)$	$T_y(m)$	$T_z(m)$	$\alpha('')$	$\beta('')$	$\gamma('')$	$dm(\times 10^{-6})$
CTLS	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208
Gauss-Helmert	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208

➢ **Statistical data by the quadratics sums of the residuals with Gauss-Helmert:**

$$\hat{\mathbf{e}}_z^T\hat{\mathbf{e}}_z = 1.015790\left(m^2\right)$$
$$\hat{\mathbf{e}}_s^T\hat{\mathbf{e}}_s = 1.015808\left(m^2\right)$$
$$\hat{\mathbf{e}}_z^T\hat{\mathbf{e}}_z + \hat{\mathbf{e}}_s^T\hat{\mathbf{e}}_s = 2.031598\left(m^2\right)$$

CTLS:

$$\hat{\mathbf{e}}_{CTLS}^T\hat{\mathbf{e}}_{CTLSP} = 1.015790\left(m^2\right)$$
$$\hat{\mathbf{e}}_{\mathbf{a}_{CTLS}}^T\hat{\mathbf{e}}_{\mathbf{a}_{CTLS}} = 1.015808\left(m^2\right)$$
$$\hat{\mathbf{e}}_{TLSP}^T\hat{\mathbf{e}}_{TLSP} + \hat{\mathbf{e}}_{\mathbf{a}_{TLSP}}^T\hat{\mathbf{e}}_{\mathbf{a}_{TLSP}} = 2.031598\left(m^2\right)$$

Comparison of the affine transformation results with CTLS and Gauss-Helmert model

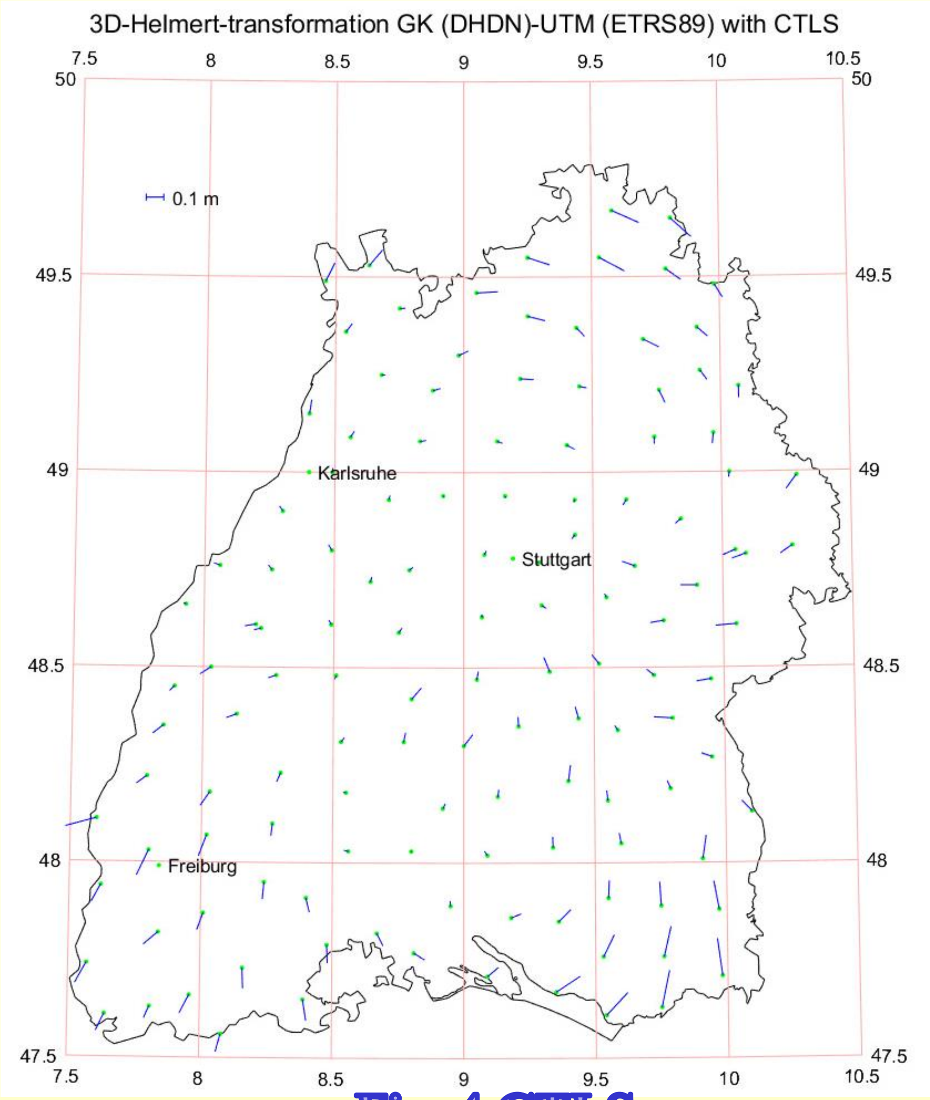

Fig. 4 CTLS

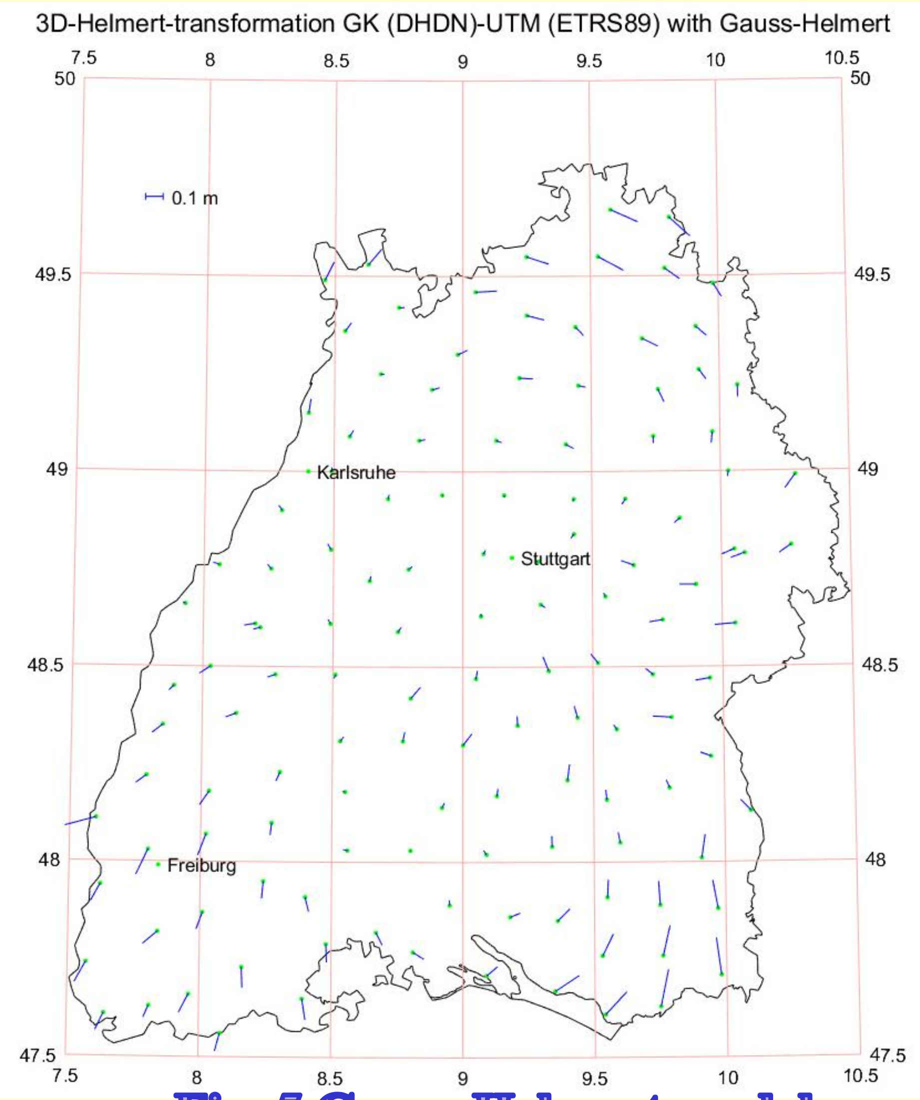

Fig. 5 Gauss-Helmert model

➧ **This identical results have also been supported by the real case study!**

5. Conclusions and further studies

- The traditional SVD method of TLS has a theoretical weakness in that it can not be applied directly when only part of the design matrix contains errors.
- The Converted Total Least Squares (CTLS) can be used to deal with stochastic design matrix in TLS problem, where the TLS problem has been successfully converted into a LS problem.
- CTLS can be easily applied with considering the weight of observations and the weight of stochastic elements of design matrix. (Completely!)
- Although the estimated transformation parameters of Partial-EIV model and CTLS are almost identical, our CTLS has its advantage without complicated iteration processing. (Efficiently!)
- This study developes one converted approach for TLS problem, which provides statistical information of parameters and stochastic design matrix, enriches the TLS algorithm, and solves the bottleneck restricting the application of TLS.
- This notable development of the CLTS reveals that CTLS estimator is identical to Gauss-Helmert model estimator in dealing with EIV models, especially in the case of coordinate transformation.
- A general connection and even identical estimates of CTLS and Gauss-Helmert model should be further studied and proved.

- ***Thank you !***

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