Complex Singular Spectrum Analysis of Earth Orientation Time Series

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Motivation

SSA — Singular Spectrum Analysis — Uni-variate time series analysis

MSSA — Multi-channel Singular Spectrum Analysis — Channels unrelated

CSSA — Complex Singular Spectrum Analysis

Earth Orientation Time Series

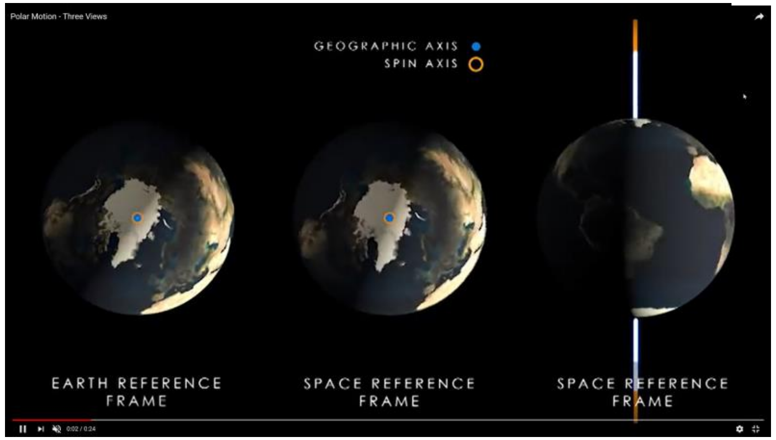
- Earth Orientation Time Series given in 1997 IERS system at 0.05 year interval has six main dataset:

  polar motion in x direction and y direction,

  Universal Time,

  Length of Day,

  celestial pole offsets of precession and nutation.



- Polar motion has three major components:

  Chandler Wobble with period about 435 days,

  Annual Wobble with nearly constant amplitude of about o.1'',

  trend.

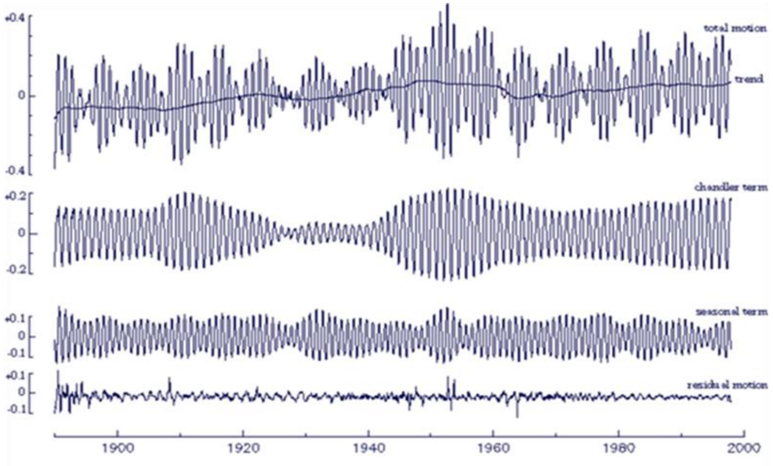


Figure II-2 X - coordinate of the pole Unit: 1".

- Polar motion time series from the year

  1960 to 2009 will be analyzed.

1. Singular Spectrum Analysis

Decomposition
- Embedding
- Singular Value Decomposition(SVD)

Reconstruction
- Grouping
- Anti-diagonal averaging

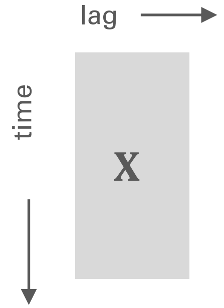
Embedding

- Consider time series $\mathrm{Y} = (x_1, x_2, x_3, \dots, x_N)$
- Create a second dimension by *lagging* the data
- The time series is repeated, but provided with a time lag in the columns
- data matrix is called *trajectory* matrix with equal values on anti-diagonals

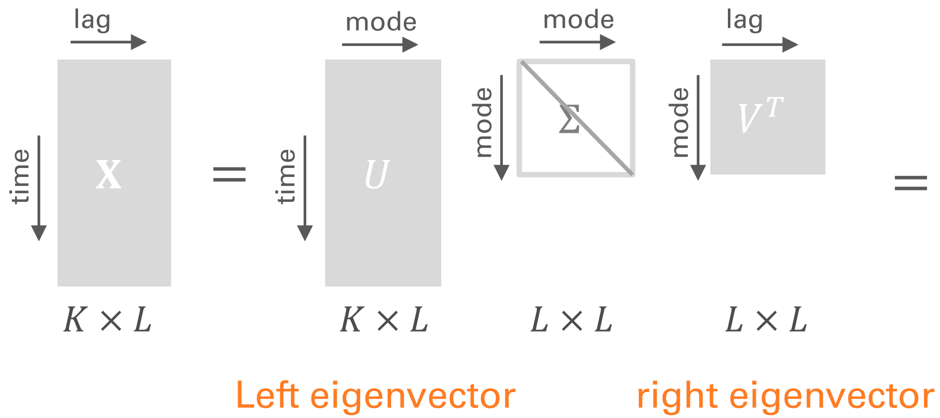
L = lag window size

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_L \\ x_2 & x_3 & x_4 & \cdots & x_{L+1} \\ x_3 & x_4 & x_5 & \cdots & x_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_K & x_{K+1} & x_{K+2} & \cdots & x_N \end{pmatrix}$$

$K = N - L + 1$ = reduced time series length



Singular Value Decomposition(SVD)



$$= \sum_{i=1}^{d} u_i \sigma_i v_i^T$$

After SVD, trajectory matrix can be written as:

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_\mathrm{d}$$

$$d = max\,\{i : \lambda_i > 0\} = rank\mathbf{X}.$$

Grouping

Splitting $\mathbf{X}$ into disjoint subsets I1,I2,...,Im as:

$$\mathbf{X} = \mathbf{X}_{\mathbf{I1}} + \mathbf{X}_{\mathbf{I2}} + \ldots + \mathbf{X}_{\mathbf{Im}}$$

Each $\mathbf{X}_i$ will reflect the properties of initial data components which have a meaningful interpretation

Anti-diagonal averaging

Note that time series $\mathrm{Y} = (x_1, x_2, x_3, \dots, x_N)$ is reconstructed by anti-diagonal averaging of $\mathbf{X}$

$$
\begin{matrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{matrix}
\qquad
\mathbf{X} = \begin{pmatrix}
x_1 & x_2 & x_3 & \cdots & x_L \\
x_2 & x_3 & x_4 & \cdots & x_{L+1} \\
x_3 & x_4 & x_5 & \cdots & x_{L+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_K & x_{K+1} & x_{K+2} & \cdots & x_N
\end{pmatrix}
$$

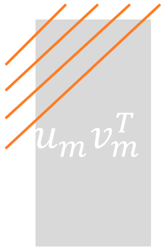
$$
x_i = \text{anti\_diag\_average}(\mathbf{X}, i) = \begin{cases}
\frac{1}{i} \sum_{j=1}^{i} \mathbf{X}_{i-j+1,j} & i = 1, \dots, L \\
\frac{1}{L} \sum_{j=1}^{L} \mathbf{X}_{i-j+1,j} & , i = L+1, \dots, K \\
\frac{1}{N-i+1} \sum_{j=i-N+L}^{L} \mathbf{X}_{i-j+1,j} & i = K+1, \dots, N
\end{cases}
$$

Anti-diagonal averaging

Now every mode can be averaged in anti-diagonal sense, resulting in the time series reconstruction.

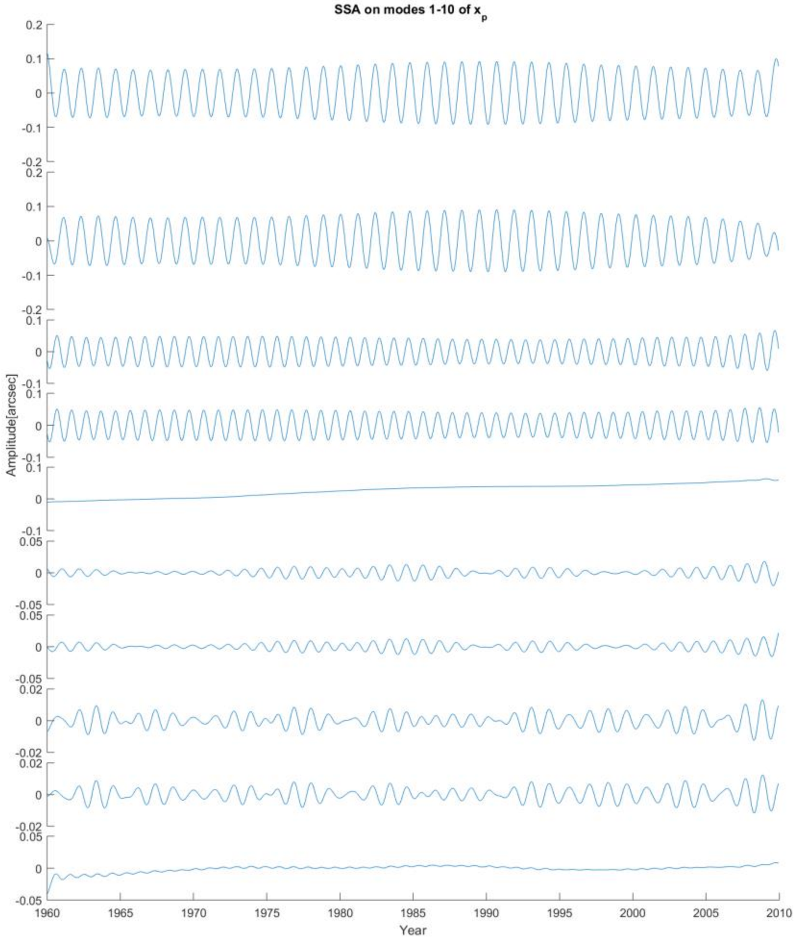
$$z_i = \mathrm{anti\_diag\_average}(u_m \sigma_m v_m^T, i)$$

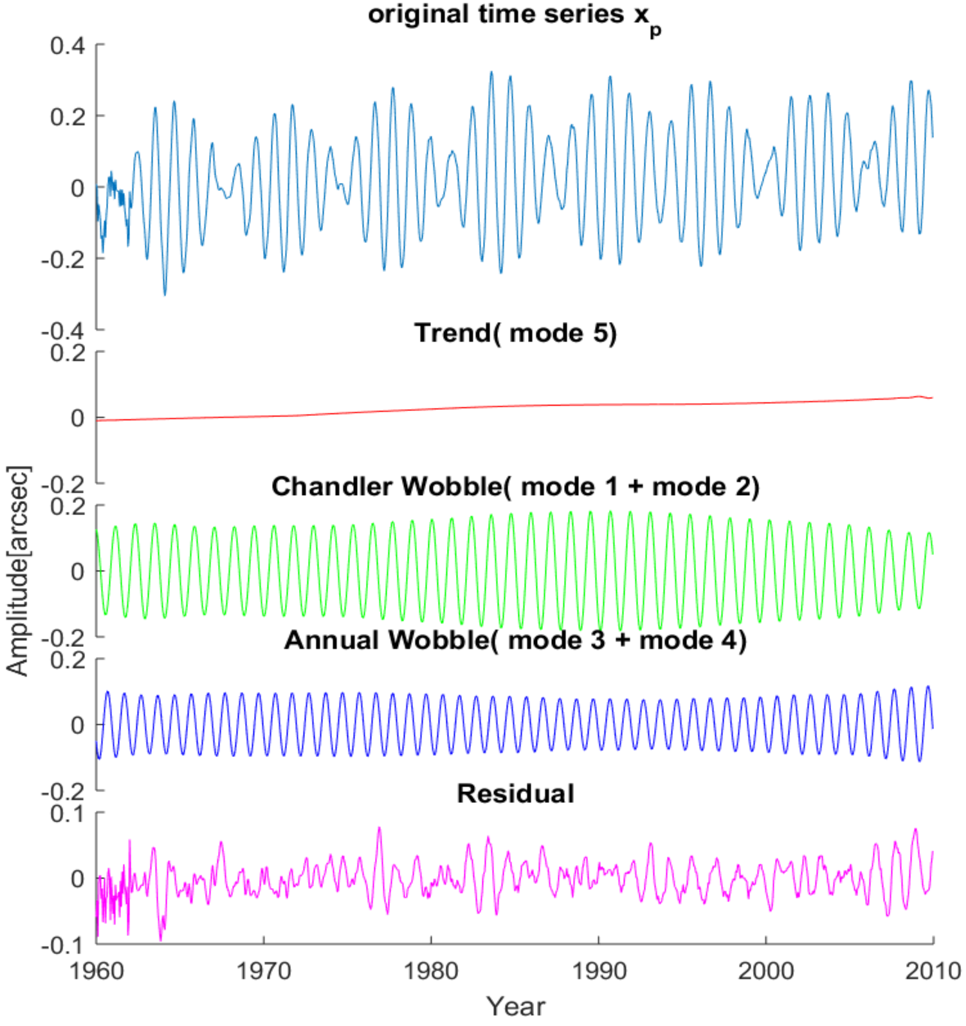
$$\mathrm{Y}_m = (z_1, z_2, z_3, \dots, z_N)$$



SSA of polar motion in x direction

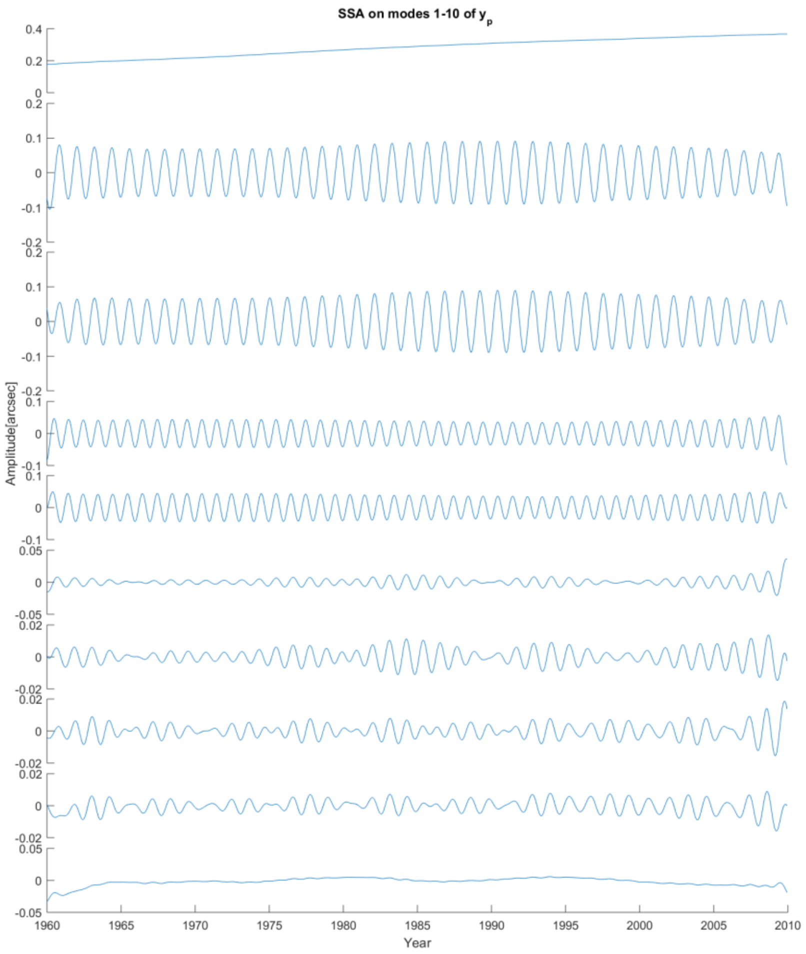
Decomposition → **Reconstruction**

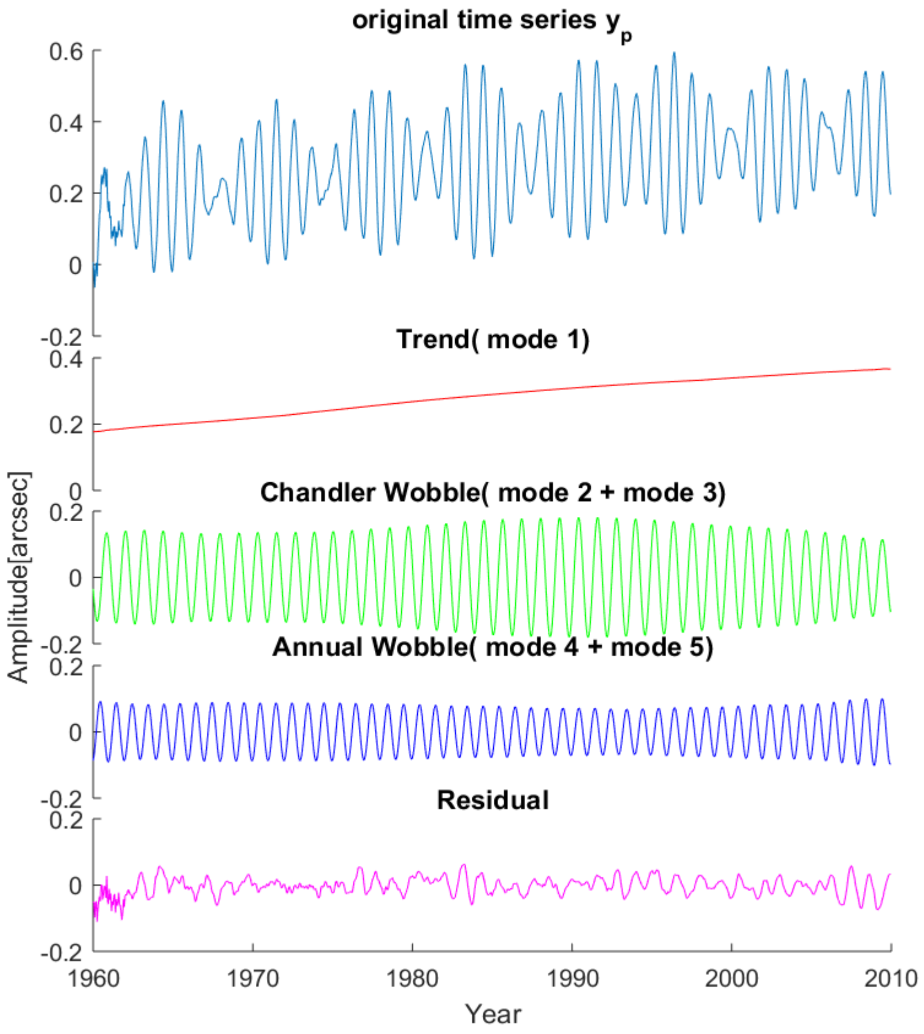




SSA of polar motion in y direction

Decomposition → **Reconstruction**





2. Multi-channel Singular Spectrum Analysis

Embedding

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Covariance matrix

↓

Reconstruction

GIS

Embedding

- Consider time series $Y = x_d\,(n)$ $(d = 1 \dots D$ and $\mathrm{n} = 1 \dots N$) be a multivariate time series with D channels of length N.
- Create a second dimension by *lagging* the data
- Data matrix is called *trajectory* matrix

lag

$L =$ lag window size

$$\mathbf{X} = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_1(L) & & x_D(1) & x_D(2) & \cdots & x_D(L) \\ x_1(2) & x_1(3) & \cdots & x_1(L+1) & & x_D(2) & x_D(3) & \cdots & x_D(L+1) \\ \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\ x_1(K) & x_1(K+1) & \cdots & x_1(N) & & x_D(K) & x_D(K+1) & \cdots & x_D(N) \end{pmatrix}$$

time

$K = N - L + 1 =$ reduced time series length

Covariance matrix and reconstruction

Covariance matrix

- Calculate the grand convariance matrix $C_{\mathbf{X}} = \frac{1}{N}\mathbf{X}^T\mathbf{X}$
- Diagonalized covariance matrix $\Lambda = Q^T C_{\mathbf{X}} Q$

  whose columns are associalated eigenvectors
- Principal components projecting trajectory matrix X onto eigenvectors.

$$A = \mathbf{X}E$$

$$a_k(n) = \sum_{d=1}^{D}\sum_{l=1}^{L} x_d(n+l-1)\, e_{dk}(l)$$

k = 1, ... DL, and n = 1, ..., N-L+1

Reconstruction

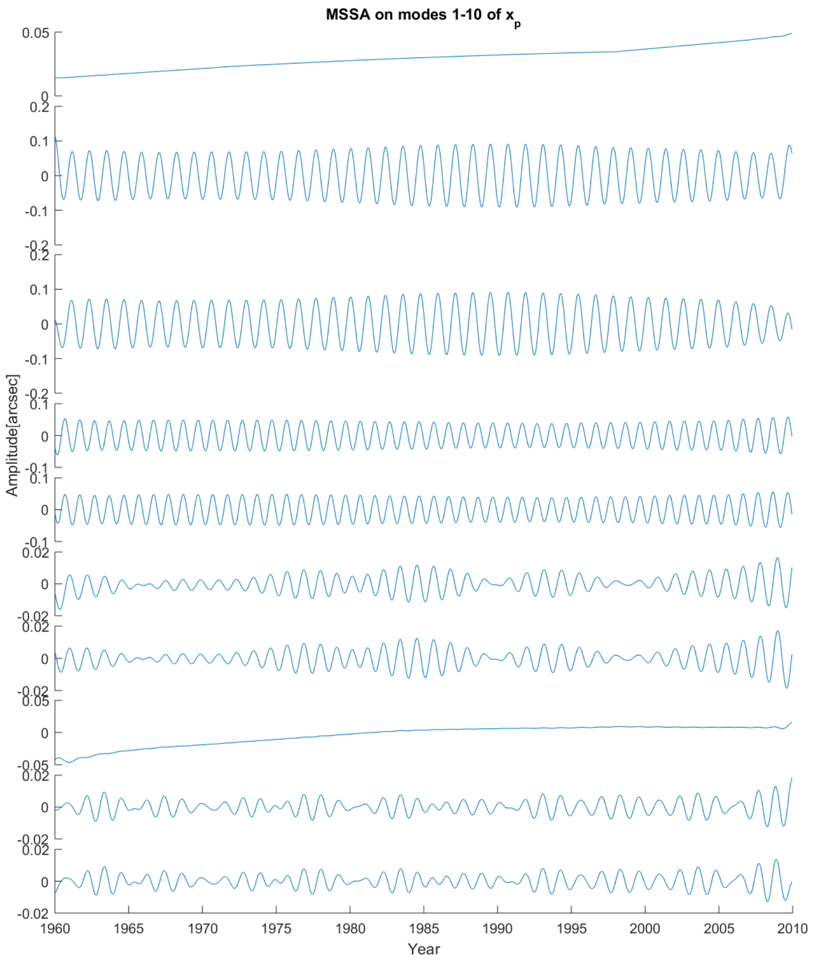
$$r_{dk}(n) = \frac{1}{M_n}\sum_{l=L_n}^{U_n} a_k(n-l+1)\, e_{dk}(l)$$

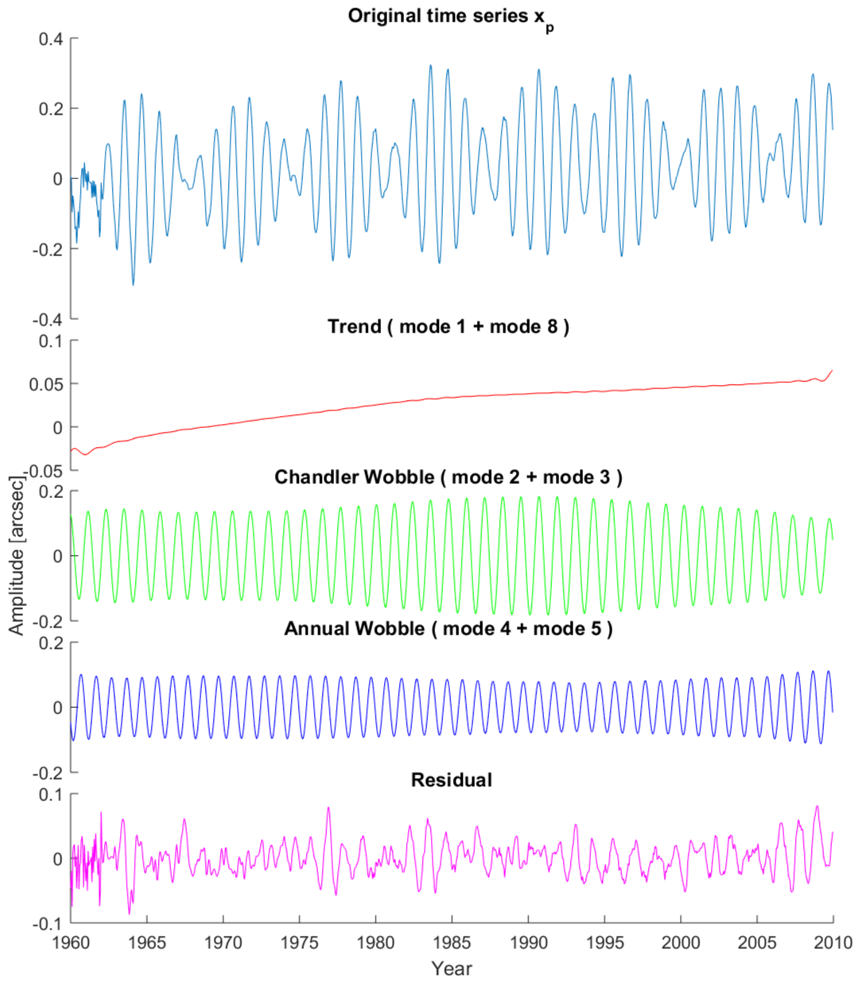
normalization factor

associatd eigenvectors at d channel

MSSA of polar motion

Decomposition → Reconstruction





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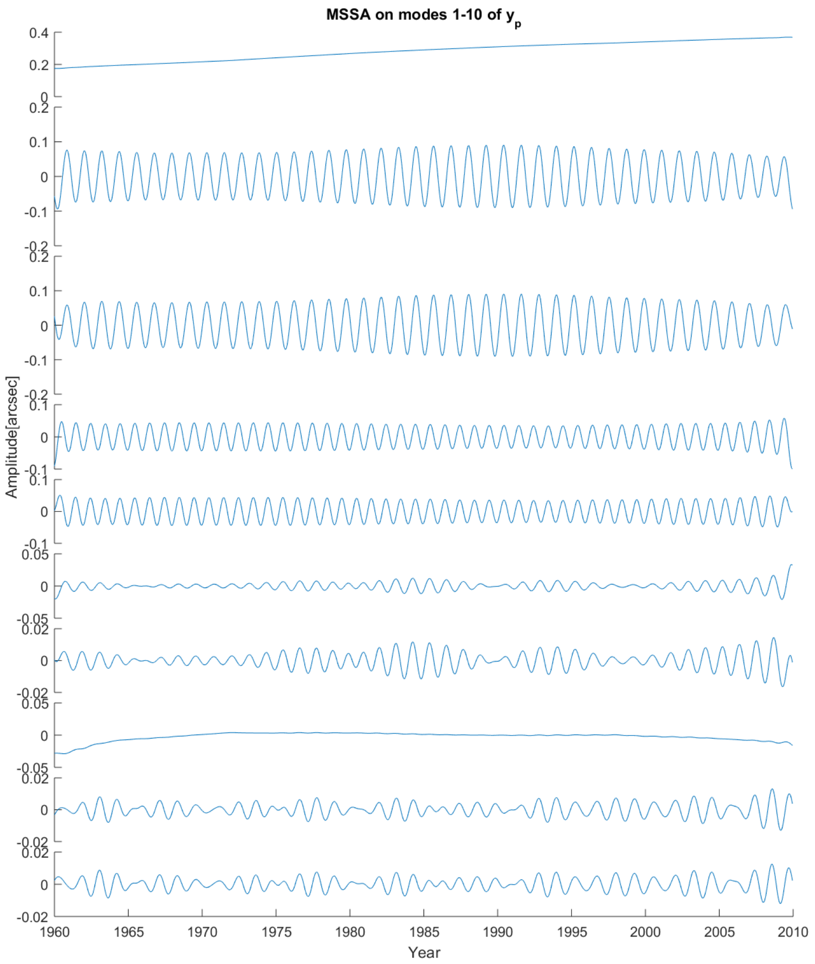
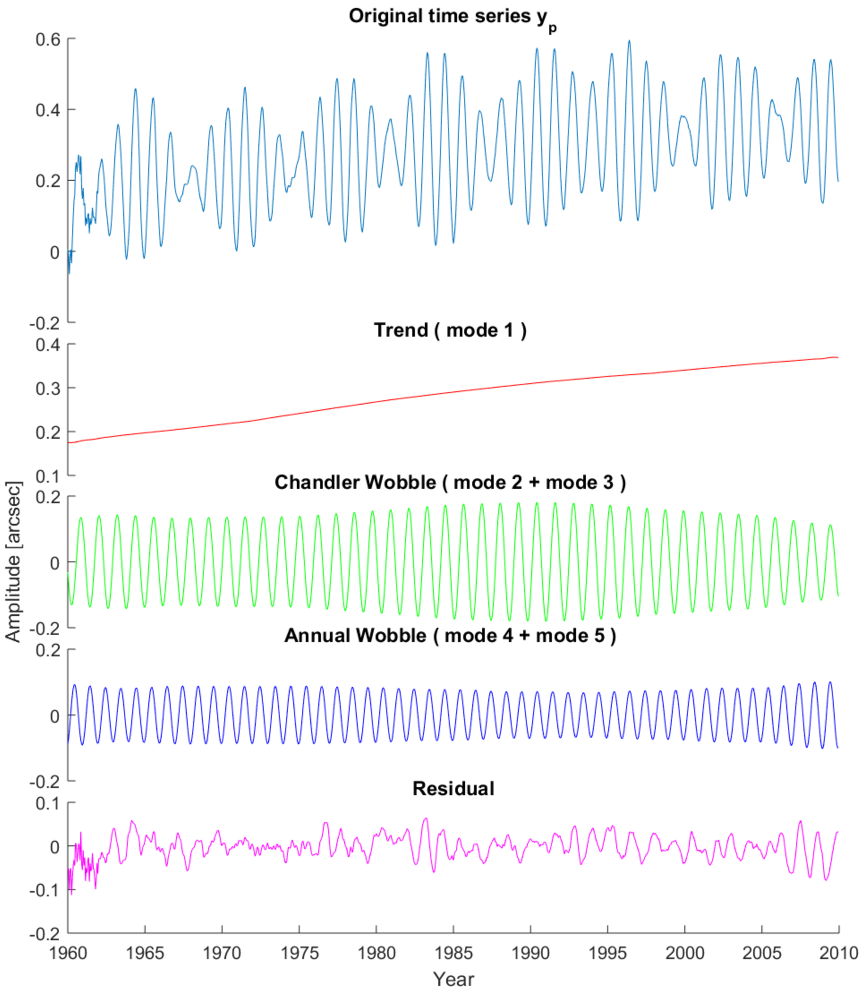
MSSA of polar motion

Decomposition

Reconstruction





3. Complex Singular Spectrum Analysis

- Generation
- Decomposition
  - Embedding
  - Singular Value Decomposition (SVD)
- Reconstruction
  - Grouping
  - Anti-diagonal averaging
- Separation
  - Separating true and imaginary part

Generation

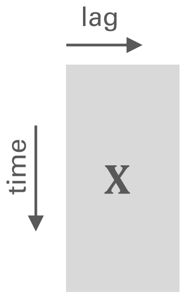
- Consider time series $Y^1 = (x_1^1, x_2^1, \dots, x_N^1)$ and $Y^2 = (x_1^2, x_2^2, \dots, x_N^2)$
- Generate a new time seires by $Y = Y^1 + i \cdot Y^2 = (x_1, x_2, \dots, x_N)$

Embedding

- Generate *trajectory* matrix

L = lag window size

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_L \\ x_2 & x_3 & x_4 & \cdots & x_{L+1} \\ x_3 & x_4 & x_5 & \cdots & x_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_N \end{pmatrix}$$



$k = N - L + 1$ = reduced time series length

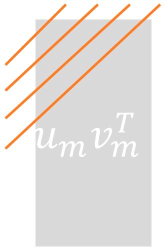
Separation

ingular Value Decomposition (SVD) → Grouping → Anti-diagonal averaging

Anti-diagonal sense can be used in every mode resulting in the time series reconstruction.

$$z_i = \text{anti\_diag\_average}(u_m \sigma_m v_m^T, i)$$

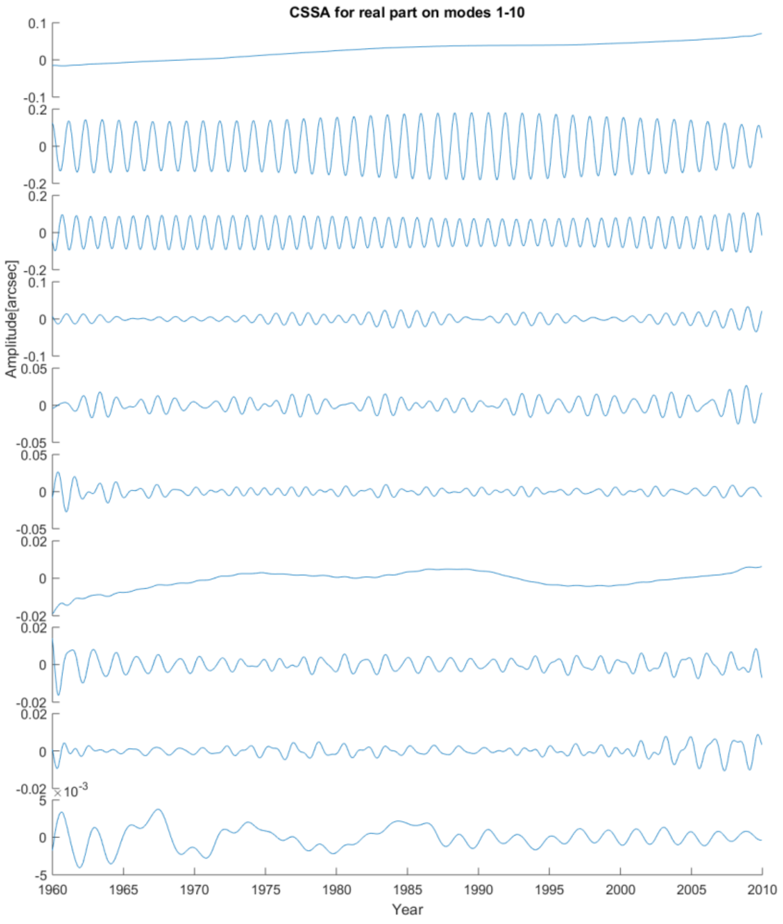
$$\mathrm{Y}_m = (z_1, z_2, z_3, \dots, z_N)$$

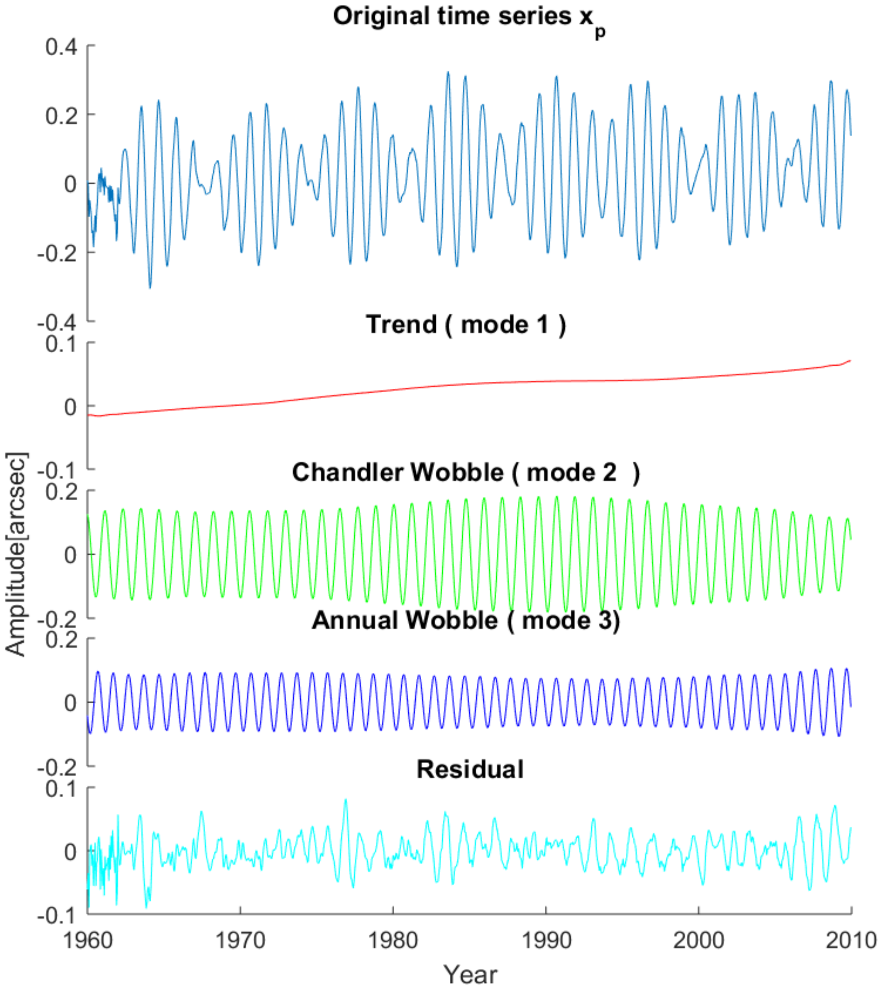


$$\mathbf{Y}_m^1 = \mathbf{real}(\mathbf{Y}_m),$$
$$\mathbf{Y}_m^2 = -\mathbf{imag}(\mathbf{Y}_m)$$

CSSA of polar motion Time Series in the real part

Decomposition → **Reconstruction**





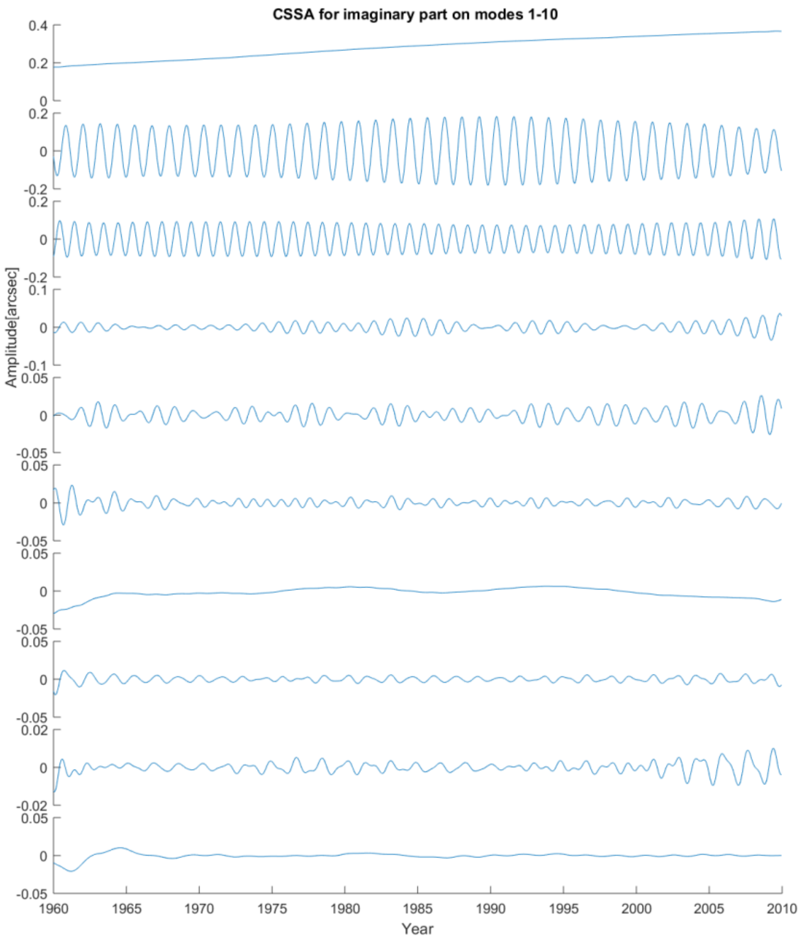
$$f = a_1 \sin(b_1 t + c_1)$$

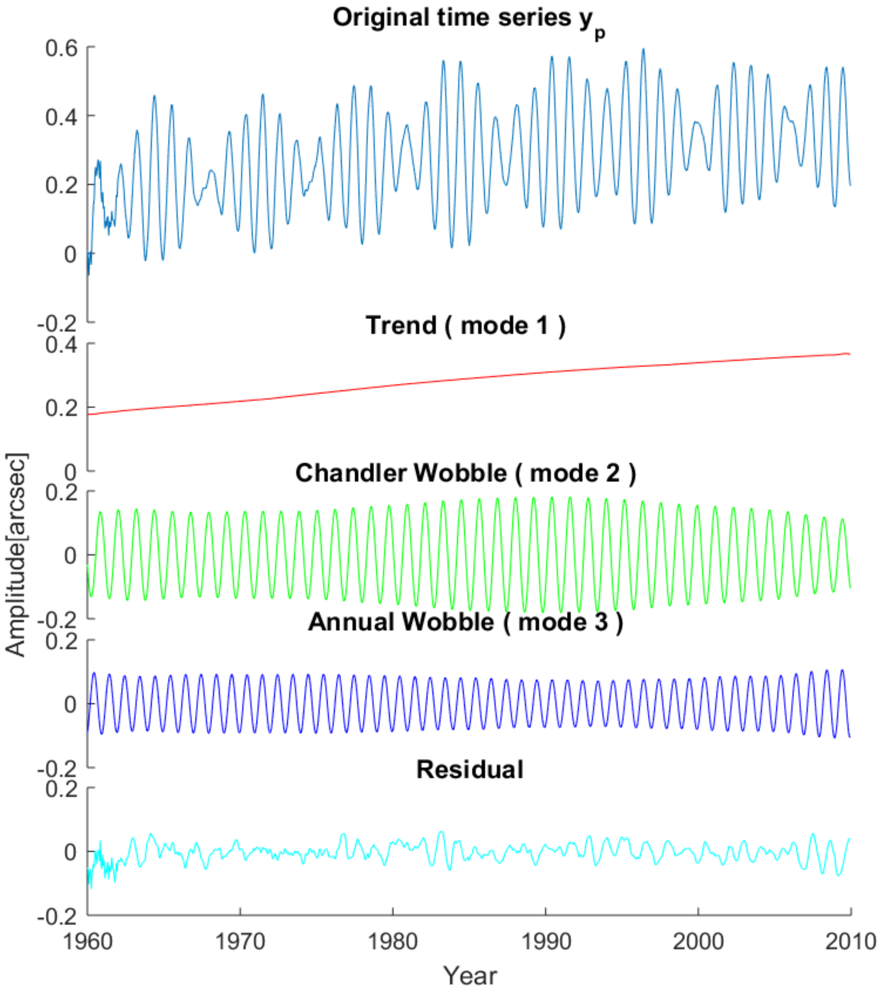
fitting parameter	a_1 [arcsec]	b_1 [rad/year]
CW	0.1517	5.305
AW	0.086 38	6.289

$$T = \frac{2\pi}{f}$$

CSSA of polar motion Time Series in imaginary part

Decomposition → Reconstruction





$$f = a_1 \sin(b_1 t + c_1)$$

fitting parameter	a_1 [arcsec]	b_1 [rad/year]
CW	0.1515	5.305
AW	0.086 35	6.29

$$T = \frac{2\pi}{f}$$

4. Comparison CSSA with SSA and MSSA

modes comparison

↓

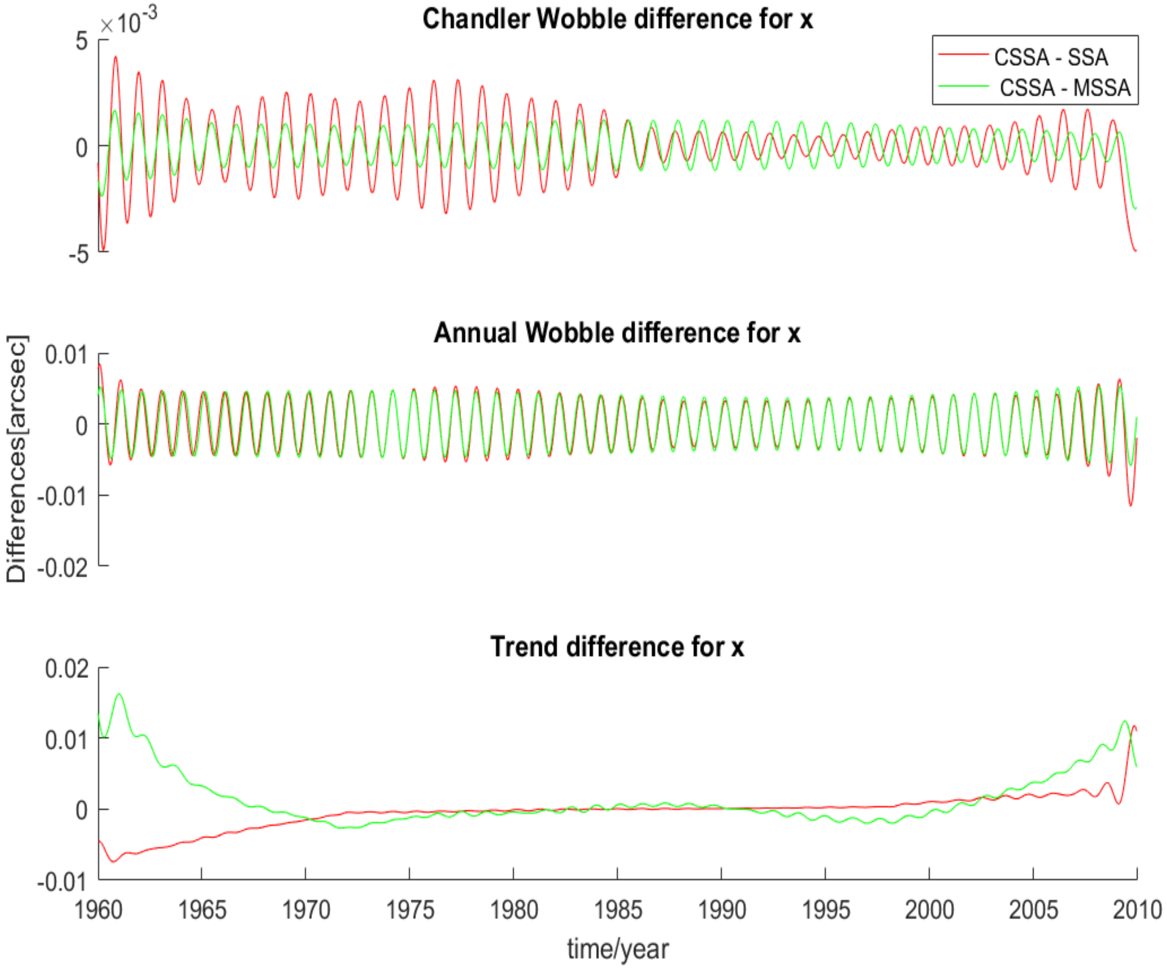
reconstruction comparison

Modes comparison

methods	SSA (for x_p)	SSA (for y_p)	MSSA (for x_p)	MSSA (for y_p)	CSSA (for x_p and y_p)
Trend	mode 5	mode 1	mode 1 and mode 8	mode 1	mode 1
Chandler Wobble	mode 1 and mode 2	mode 2 and mode 3	mode 2 and mode 3	mode 2 and mode 3	mode 2
Annual Wobble	mode 3 and mode 4	mode 4 and mode 5	mode 4 and mode 5	mode 4 and mode 5	mode 3
Residual	else	else	else	else	else

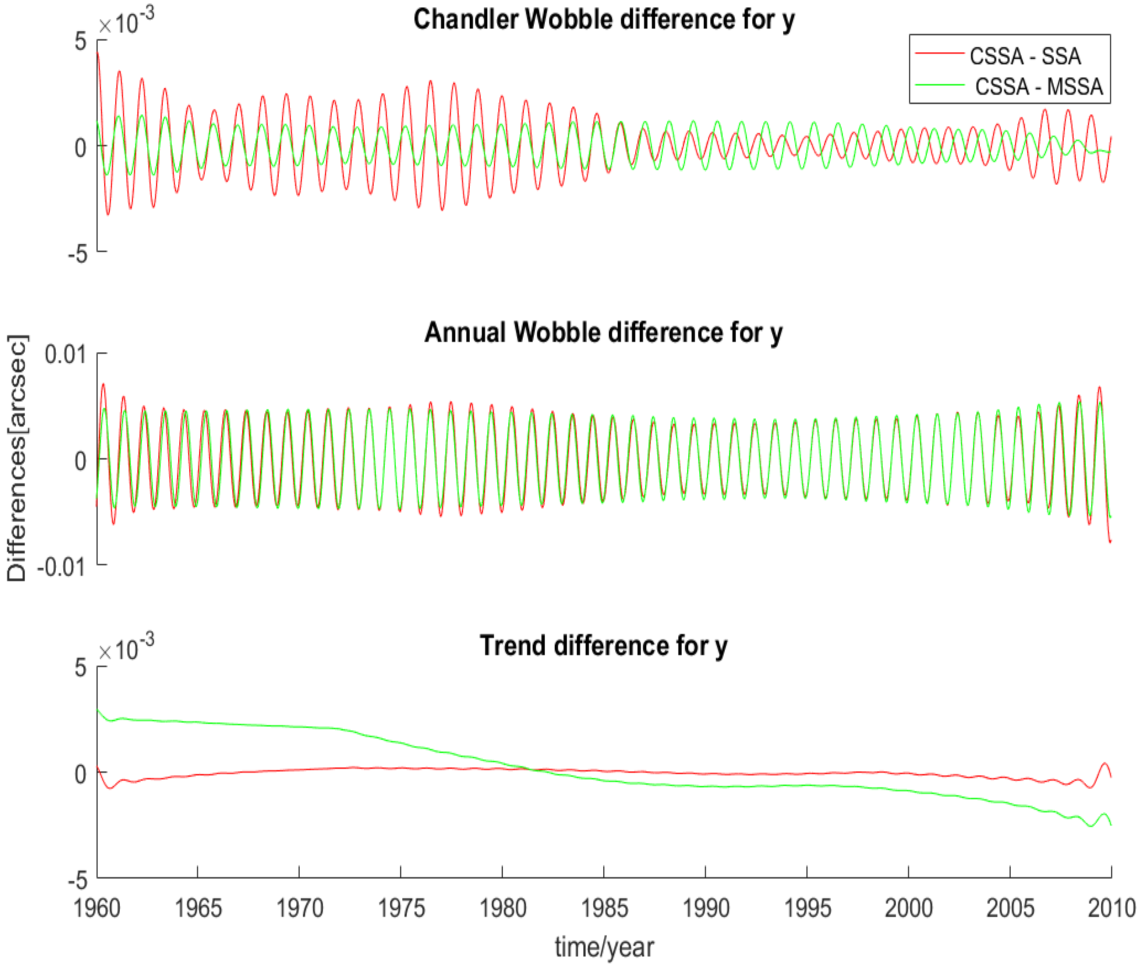
GIS

main components comparison in x direction



RMS for differences	CW	AW	trend
CSSA-SSA	0.0015	0.0033	0.0024
CSSA-MSSA	$8.0749e-04$	0.0031	0.0040

main components comparison in y direction



RMS for differences	CW	AW	trend
CSSA-SSA	0.0013	0.0032	$2.1443e-04$
CSSA-MSSA	$7.1005e-04$	0.0031	0.0015

5. Conclusion

- CSSA with constant single mode in grouping
- CSSA perform well in decomposition Earth Orientation Time Series into Chandler Wobble, Annual Wobble and trend.

6. Outlook

- CSSA can be just used in 2D data. The next step of researth is to find out the usefulness in multi-channel time series.
- CSSA shows advantage in polar motion time series, in the future, it may use in other fields.

7. Reference

- H. Hassani(2007). “Singular Spectrum Analysis: Methodology and Comparison”. In: Journal of Data Science, pp. 239–257.
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Thank you