Complex Singular Spectrum Analysis of Earth Orientation Time Series

Yang Li, University of Stuttgart

Supervisor: Prof. Dr.-Ing. Nico Sneeuw, Institute of Geodesy, University of Stuttgart
Prof. Dr. Weiping Jiang, GNSS Research Center, Wuhan University
Motivation

- **SSA**
  - Singular Spectrum Analysis
  - Uni-variate time series analysis

- **MSSA**
  - Multi-channel Singular Spectrum Analysis
  - Channels unrelated

- **CSSA**
  - Complex Singular Spectrum Analysis
Earth Orientation Time Series

- Earth Orientation Time Series given in 1997 IERS system at 0.05 year interval has six main dataset:
  - polar motion in x direction and y direction,
  - Universal Time,
  - Length of Day,
  - celestial pole offsets of precession and nutation.

- Polar motion has three major components:
  - Chandler Wobble with period about 435 days,
  - Annual Wobble with nearly constant amplitude of about $0.1''$ trend.

- Polar motion time series from the year 1960 to 2009 will be analyzed.
1. **Singular Spectrum Analysis**

- **Decomposition**
  - Singular Value Decomposition (SVD)
  - Embedding

- **Reconstruction**
  - Grouping
  - Anti-diagonal averaging
Consider time series \( Y = (x_1, x_2, x_3, \ldots, x_N) \)

Create a second dimension by *lagging* the data

The time series is repeated, but provided with a time lag in the columns

Data matrix is called *trajectory* matrix with equal values on anti-diagonals

\[
X = \begin{pmatrix}
  x_1 & x_2 & x_3 & \cdots & x_L \\
  x_2 & x_3 & x_4 & \cdots & x_{L+1} \\
  x_3 & x_4 & x_5 & \cdots & x_{L+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_K & x_{K+1} & x_{K+2} & \cdots & x_N \\
\end{pmatrix}
\]

\( L = \) lag window size

\( K = N - L + 1 = \) reduced time series length
Singular Value Decomposition (SVD)

\[ \mathbf{X} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \]

Left eigenvector \( \mathbf{X} \)  
Right eigenvector \( \mathbf{U}, \mathbf{\Sigma}, \mathbf{V}^T \)

After SVD, trajectory matrix can be written as:

\[ \mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_d \]

\[ d = \max \{ i : \lambda_i > 0 \} = \text{rank} \mathbf{X} \]
Grouping

Splitting $X$ into disjoint subsets $I_1, I_2, \ldots, I_m$ as:

$$X = X_{I_1} + X_{I_2} + \ldots + X_{I_m}$$

Each $X_i$ will reflect the properties of initial data components which have a meaningful interpretation.
Note that time series \( Y = (x_1, x_2, x_3, ..., x_N) \) is reconstructed by anti-diagonal averaging of \( X \)

\[
X = \begin{pmatrix}
x_1 & x_2 & x_3 & \cdots & x_L \\
x_2 & x_3 & x_4 & \cdots & x_{L+1} \\
x_3 & x_4 & x_5 & \cdots & x_{L+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_K & x_{K+1} & x_{K+2} & \cdots & x_N
\end{pmatrix}
\]

\[
x_i = \text{anti_diag_average}(X, i) = \begin{cases} 
\frac{1}{L} \sum_{j=i}^{i+L} x_{i-j+1,j} & i = 1, \ldots, L \\
\frac{1}{L} \sum_{j=i}^{i+L} x_{i-j+1,j} & i = L + 1, \ldots, K \\
\frac{1}{N-L+1} \sum_{j=i-N+L}^{i+1} x_{i-j+1,j} & i = K + 1, \ldots, N
\end{cases}
\]
Anti-diagonal averaging

Now every mode can be averaged in anti-diagonal sense, resulting in the time series reconstruction.

\[ z_i = \text{anti\_diag\_average}(u_m \sigma_m v_m^T, i) \]

\[ Y_m = (z_1, z_2, z_3, ..., z_N) \]
SSA of polar motion in x direction

Decomposition

Reconstruction

Decomposition

Reconstruction

original time series $x_p$

Trend (mode 5)

Chandler Wobble (mode 1 + mode 2)

Annual Wobble (mode 3 + mode 4)

Residual

$\text{SSA on modes 1-10 of } x_x$

curve fitting by equation $f = a_1 \sin(b_1 t + c_1)$
SSA of polar motion in y direction

Decomposition

Reconstruction

Curve fitting by equation $f = a_1 \sin(b_1 t + c_1)$

Decomposition

Reconstruction

Original time series $y_p$

Trend (mode 1)

Chandler Wobble (mode 2 + mode 3)

Annual Wobble (mode 4 + mode 5)

Residual
2. Multi-channel Singular Spectrum Analysis

- Embedding
- Covariance matrix
- Reconstruction
Embedding

- Consider time series $Y = x_d(n)$ ($d = 1 \ldots D$ and $n = 1 \ldots N$) be a multivariate time series with $D$ channels of length $N$.
- Create a second dimension by *lagging* the data
- Data matrix is called *trajectory* matrix

$$X = \begin{pmatrix}
x_1(1) & x_1(2) & \cdots & x_1(L) & x_D(1) & x_D(2) & \cdots & x_D(L) \\
x_1(2) & x_1(3) & \cdots & x_1(L+1) & x_D(2) & x_D(3) & \cdots & x_D(L+1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_1(K) & x_1(K+1) & \cdots & x_1(N) & x_D(K) & x_D(K+1) & \cdots & x_D(N)
\end{pmatrix}
$$

$L = \text{lag window size}$

$$K = N - L + 1 = \text{reduced time series length}$$
Covariance matrix and reconstruction

Covariance matrix

- Calculate the grand covariance matrix \( C_X = \frac{1}{N}X^TX \)
- Diagonalized covariance matrix \( \Lambda = Q^T C_X Q \)
- Principal components projecting trajectory matrix \( X \) onto eigenvectors.

Reconstruction

\[
A = XE
\]

\[
a_k(n) = \sum_{d=1}^{D} \sum_{l=1}^{L} x_d(n + l - 1)e_{dk}(l)
\]

\[
r_{dk}(n) = \frac{1}{M_n} \sum_{n=1}^{V_n} a_k(n - l + 1)e_{dk}(l)
\]

\( k = 1, \ldots, DL \), and \( n = 1, \ldots, N-L+1 \)

associated eigenvectors at \( d \) channel

normalization factor
MSSA of polar motion

Decomposition

Reconstruction

MSSA on modes 1-10 of $x_p$

Original time series $x_p$

Trend (mode 1 + mode 8)

Chandler Wobble (mode 2 + mode 3)

Annual Wobble (mode 4 + mode 5)

Residual
MSSA of polar motion

Decomposition

Reconstruction

\[ f(t) = a_1 \sin(b_1 t + c_1) \]
3. Complex Singular Spectrum Analysis

- Generation
- Decomposition
  - Embedding
  - Singular Value Decomposition (SVD)
- Reconstruction
  - Grouping
  - Anti-diagonal averaging
- Separation
  - Separating true and imaginary part
Generation

- Consider time series $Y^1 = (x_1^1, x_2^1, ..., x_N^1)$ and $Y^2 = (x_1^2, x_2^2, ..., x_N^2)$
- Generate a new time series by $Y = Y^1 + i \cdot Y^2 = (x_1, x_2, ..., x_N)$

Embedding

- Generate trajectory matrix

$$ X = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_L \\ x_2 & x_3 & x_4 & \cdots & x_{L+1} \\ x_3 & x_4 & x_5 & \cdots & x_{L+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_N \end{pmatrix} $$

$L = \text{lag window size}$

$k = N - L + 1 = \text{reduced time series length}$
Anti-diagonal sense can be used in every mode resulting in the time series reconstruction.

\[ z_i = \text{anti\_diag\_average}(u_m \sigma_m v_{m}^T, i) \]

\[ Y_m = (z_1, z_2, z_3, ..., z_N) \]

\[ Y_m^1 = \text{real}(Y_m), \]
\[ Y_m^2 = -\text{imag}(Y_m) \]
CSSA of polar motion Time Series in the real part

Decomposition

Reconstruction

\[ f = a_1 \sin(b_1 t + c_1) \]

<table>
<thead>
<tr>
<th>fitting parameter</th>
<th>(a_1) [arcsec]</th>
<th>(b_1) [rad/year]</th>
</tr>
</thead>
<tbody>
<tr>
<td>CW</td>
<td>0.1517</td>
<td>5.305</td>
</tr>
<tr>
<td>AW</td>
<td>0.08638</td>
<td>6.289</td>
</tr>
</tbody>
</table>

\[ T = \frac{2\pi}{f} \]
CSSA of polar motion Time Series in imaginary part

Decomposition

Reconstruction

\[ f = a_1 \sin(b_1 t + c_1) \]

<table>
<thead>
<tr>
<th>fitting parameter</th>
<th>( a_1 ) [arcsec]</th>
<th>( b_1 ) [rad/year]</th>
</tr>
</thead>
<tbody>
<tr>
<td>CW</td>
<td>0.1515</td>
<td>5.305</td>
</tr>
<tr>
<td>AW</td>
<td>0.08635</td>
<td>6.29</td>
</tr>
</tbody>
</table>

\[ T = \frac{2\pi}{f} \]
4. *Comparison CSSA with SSA and MSSA*

- modes comparison
- reconstruction comparison
## Modes comparison

<table>
<thead>
<tr>
<th>methods</th>
<th>SSA (for $x_p$)</th>
<th>SSA (for $y_p$)</th>
<th>MSSA (for $x_p$)</th>
<th>MSSA (for $y_p$)</th>
<th>CSSA (for $x_p$ and $y_p$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trend</td>
<td>mode 5</td>
<td>mode 1</td>
<td>mode 1 and mode 8</td>
<td>mode 1</td>
<td>mode 1</td>
</tr>
<tr>
<td>Chandler Wobble</td>
<td>mode 1 and mode 2</td>
<td>mode 2 and mode 3</td>
<td>mode 2 and mode 3</td>
<td>mode 2</td>
<td>mode 2</td>
</tr>
<tr>
<td>Annual Wobble</td>
<td>mode 3 and mode 4</td>
<td>mode 4 and mode 5</td>
<td>mode 4 and mode 5</td>
<td>mode 4</td>
<td>mode 3</td>
</tr>
<tr>
<td>Residual</td>
<td>else</td>
<td>else</td>
<td>else</td>
<td>else</td>
<td>else</td>
</tr>
</tbody>
</table>
main components comparison in x direction

<table>
<thead>
<tr>
<th>RMS for differences</th>
<th>CW</th>
<th>AW</th>
<th>trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSSA-SSA</td>
<td>0.0015</td>
<td>0.0033</td>
<td>0.0024</td>
</tr>
<tr>
<td>CSSA-MSSA</td>
<td>8.0749e-04</td>
<td>0.0031</td>
<td>0.0040</td>
</tr>
</tbody>
</table>
main components comparison in y direction

<table>
<thead>
<tr>
<th>RMS for differences</th>
<th>CW</th>
<th>AW</th>
<th>trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSSA-SSA</td>
<td>0.0013</td>
<td>0.0032</td>
<td>2.1443e-04</td>
</tr>
<tr>
<td>CSSA-MSSA</td>
<td>7.1005e-04</td>
<td>0.0031</td>
<td>0.0015</td>
</tr>
</tbody>
</table>
5. Conclusion

• CSSA with constant single mode in grouping
• CSSA perform well in decomposition Earth Orientation Time Series into Chandler Wobble, Annual Wobble and trend.
6. **Outlook**

- CSSA can be just used in 2D data. The next step of research is to find out the usefulness in multi-channel time series.
- CSSA shows advantage in polar motion time series, in the future, it may use in other fields.
7. Reference


Thank you