Implementation of the Sea-Level-Equation

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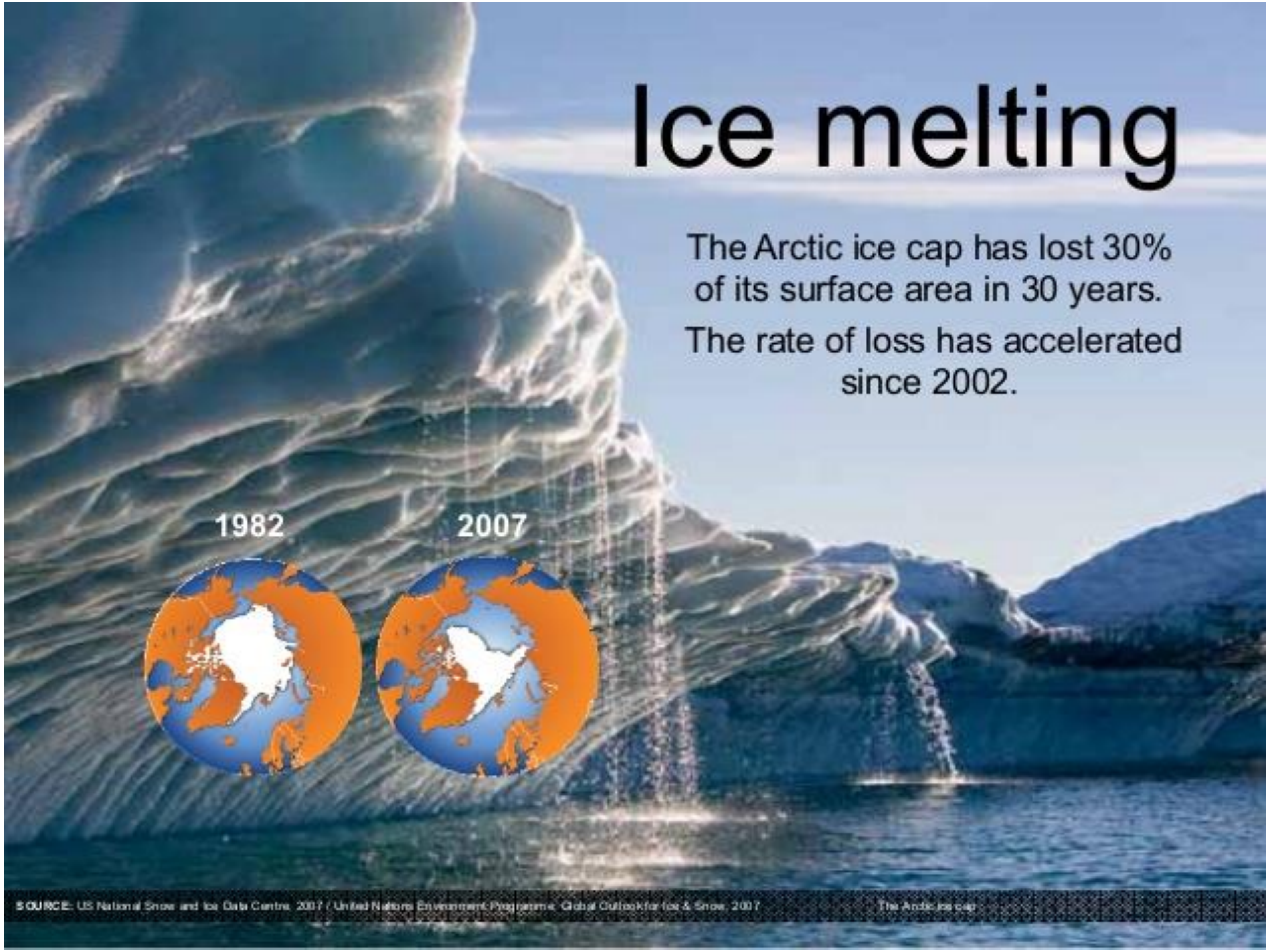


https://www.slideshare.net/AndersLindgren4u/whats-wrong-with-a-little-climate-change-2026882/49-Ice_meltingLakes_are_beginning_to

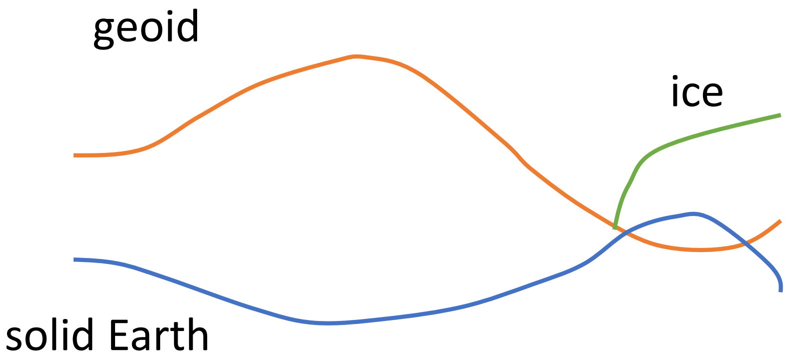
Definition of Sea-level

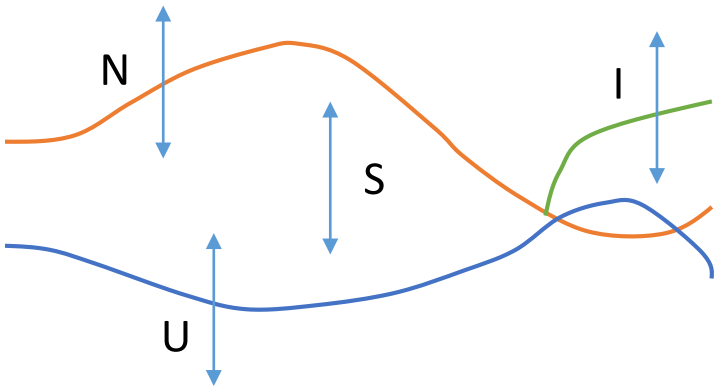
- Sea-Level is defined as difference between geoid and solid surface of the Earth
- Sea-Level change (S) is the difference between geoid variation (N) and vertical displacement of the solid Earth (U)

$$S = N - U$$

- Ice Thickness Variation

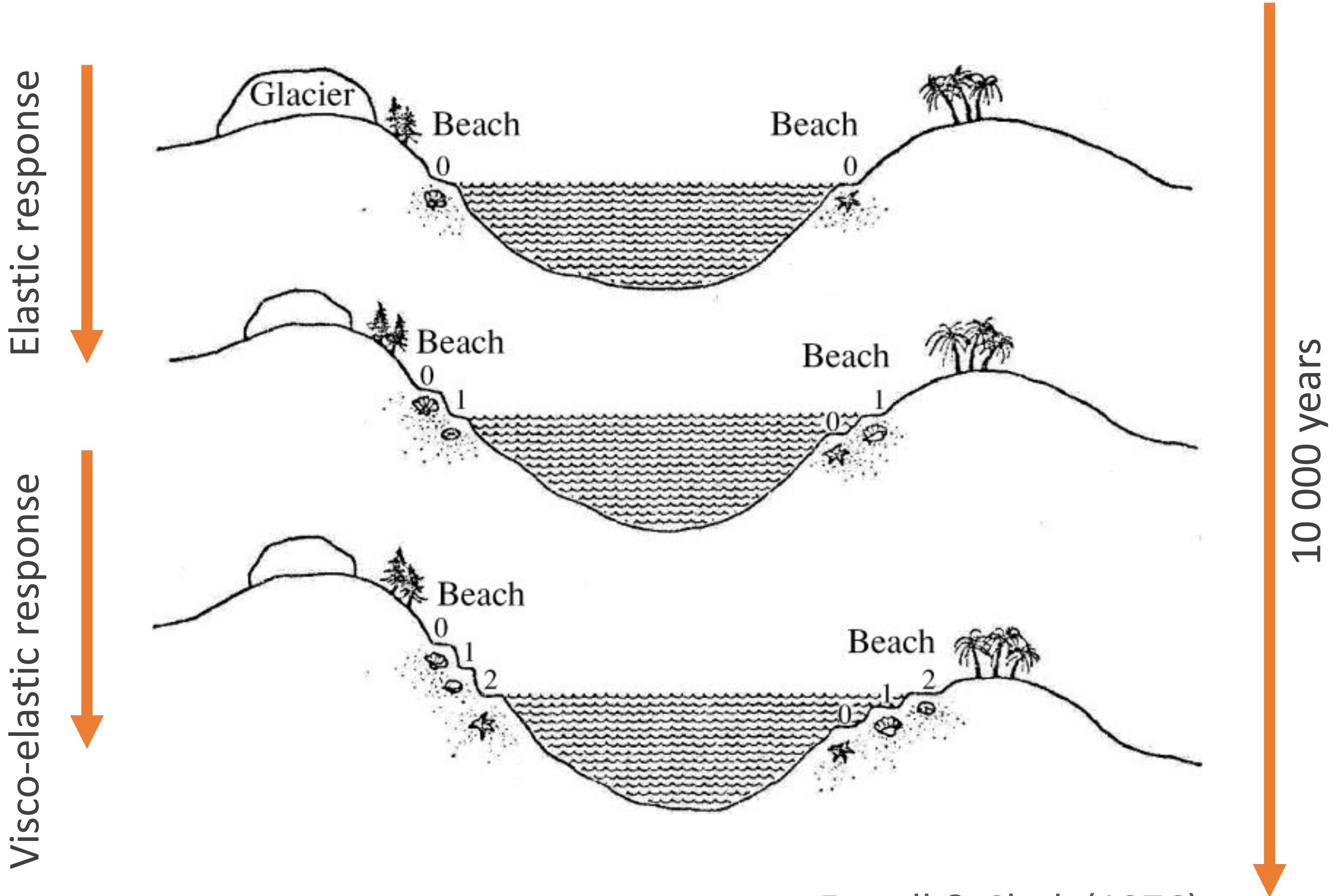
$$I = T(t_j) - T(t_{j+1})$$





Effects

- Deformation of the Earth
  - Elastic
  - Visco-elastic
- Rotation of the Earth
- Change of shorelines



Farrell & Clark (1976)

Analytical Formulation of the Sea-Level Equation

- 2 main principles
  - Principle of mass conservation → mass of the melting Ice is equal to the mass of added water
  - Water is equipotential surface → before and after adding the water, the sea surface is an equipotential surface
- Use Bruns's formula to calculate geoid variation (N)

$$N = \frac{\Phi}{\gamma} + c$$

- Φ is the disturbing potential, γ is the normal gravity and c is the potential anomaly
- Disturbing potential induced: disturbing affect to water redistribution
- In the following:
  - Fixed shorelines
  - Spherical-symmetric non-rotating elastic Earth

Analytical Formulation of the Sea-Level Equation

- Sea-level change: $S = \frac{\Phi}{\gamma} + c - U$
- Determine constant c → use principle of mass conservation with mass of melting ice M_I and mass of water M_W

$$M_I + M_W = 0$$

$$M_I + \rho_W \int_O S \, dA = M_I + \rho_W \int_O (N - U) \, dA = M_I + \rho_W A_O c + \rho_W \int_O \left(\frac{\Phi}{\gamma} - U \right) dA = 0$$

$$c = -\frac{M_I}{\rho_W A_O} - \frac{1}{A_O} \int_O \left(\frac{\Phi}{\gamma} - U \right) dA$$

Analytical Formulation of the Sea-Level Equation

- Using this the sea-level is given by

$$S = \frac{\Phi}{\gamma} - U - \frac{M_I}{\rho_W A_O} - \frac{1}{A_O}\int_O \left(\frac{\Phi}{\gamma} - U\right) dA$$

- dA is the element of area on a sphere $dA = R^2 \cos\varphi \; d\varphi d\lambda$
- The eustatic sea level change is

$$S^E = -\frac{M_I}{\rho_W A_O}$$

- Can be used for a first guess→ sea level change is equal at all locations

Analytical Formulation of the Sea-Level Equation

- disturbing potential include gravitational potential of ice-load and water-load

$$v_{\text{load},I} = G\rho_I \iint \frac{I(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} dA$$

$$v_{\text{load},W} = G\rho_W \iint \mathcal{O}(\boldsymbol{x}') \frac{S(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} dA$$

- Where $\mathcal{O}$ is the ocean function

$$\mathcal{O}(\boldsymbol{x}) = \begin{Bmatrix} 1 \text{ if } \boldsymbol{x} \in \text{oceans} \\ 0 \text{ if } \boldsymbol{x} \in \text{land} \end{Bmatrix}, \boldsymbol{x} \in \Omega$$

Analytical Formulation of the Sea-Level Equation

- Integral can be written as a convolution on the sphere using Green functions
- Structure of Sea-Level-Equation is analogous to Fredholm equation of second kind

$$u(x) = f(x) + \lambda \int_a^b K(x, x')u(x')dx'$$

- G_S is the sea level Green Function

$$\frac{G_S}{\gamma} = \frac{G_\Phi}{\gamma} - G_U$$

Analytical Formulation of the Sea-Level Equation

- Green function depends on load coefficients h and k
- As convolution it can be written as

$$\Phi = \rho_I G_\Phi \otimes_I I + \rho_w G_\Phi \otimes_O S$$

$$\mathrm{U} = \rho_I G_\mathrm{U} \otimes_I I + \rho_w G_\mathrm{U} \otimes_O S$$

- Using a convolution expression the sea level equation is given by

$$S = \frac{\rho_i}{\gamma} G_S \otimes_i I + \frac{\rho_W}{\gamma} G_S \otimes_O S - \frac{M_I}{\rho_W A_O} - \frac{\rho_I}{\gamma} \overline{G_S \otimes_i I} - \frac{\rho_W}{\gamma} \overline{G_S \otimes_O S}$$

Numerical Formulation of the Sea-Level Equation (Pseudo-spectral approach)

- The unknown sea level change on both sides of the equation
- → iterative solution

$$S^0 = S^E$$

$$S^1 = \frac{\rho_i}{\gamma} G_S \otimes_i I + \frac{\rho_W}{\gamma} G_S \otimes_O S^0 - \frac{M_I}{\rho_W A_O} - \frac{\rho_I}{\gamma} \overline{G_S \otimes_i I} - \frac{\rho_W}{\gamma} \overline{G_S \otimes_O S^0}$$

$$S^2 = \frac{\rho_i}{\gamma} G_S \otimes_i I + \frac{\rho_W}{\gamma} G_S \otimes_O S^1 - \frac{M_I}{\rho_W A_O} - \frac{\rho_I}{\gamma} \overline{G_S \otimes_i I} - \frac{\rho_W}{\gamma} \overline{G_S \otimes_O S^1}$$

$$\dots$$

$$S^t = \frac{\rho_i}{\gamma} G_S \otimes_i I + \frac{\rho_W}{\gamma} G_S \otimes_O S^{t-1} - \frac{M_I}{\rho_W A_O} - \frac{\rho_I}{\gamma} \overline{G_S \otimes_i I} - \frac{\rho_W}{\gamma} \overline{G_S \otimes_O S^{t-1}}$$

Numerical Formulation of the Sea-Level Equation (Pseudo-spectral approach)

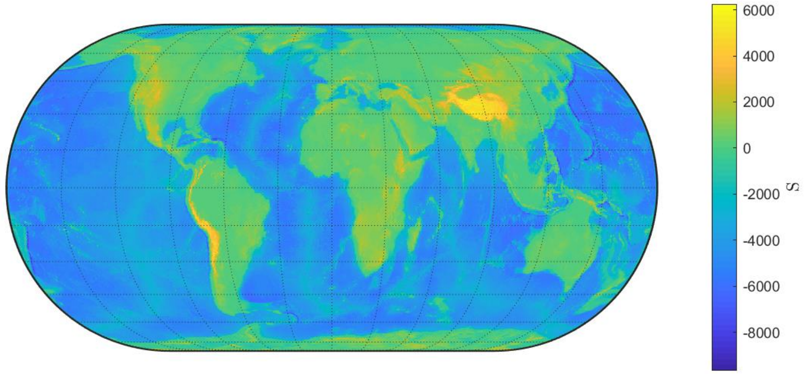
- For numerical formulation spherical harmonics are used
- Describe input field’s of Ice, Ocean, Topography in spherical harmonic coefficients → using *SH-BUNDLE*
- Spherical harmonic expression of the Green functions

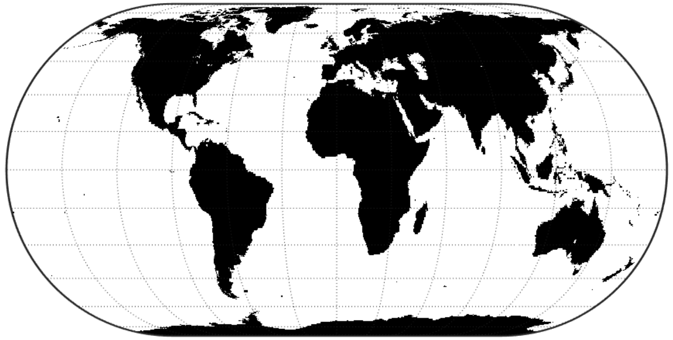
Implementation

- Implementation in Matlab
- 2 Algorithms are used
  - Mitrovica
  - Spada
- 0.5 degree grid resolution
- spherical harmonics up to degree $l_{\max} = 140$

Data

- Variation of Ice thickness → ICE-5G model
  - Global ice sheet reconstruction
  - 1 degree resolution
  - 21 000 years, every 500/ 1000 years
- Load coefficient (h_l, k_l) → PREM-Model
- Topography → ETOPO1 Model from NOAA
- Ocean-function → can be calculated from Topography





Some results

Conclusion and outlook

- Implement more effects for more realistic results
  - No fixed shorelines
  - Rotating Earth
  - Viscoelastic Earth model
- Estimate ice for future years

Thank you for your attention!

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References

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