Second Workshop of DAAD Thematic Network
Modern Geodetic Space Techniques for Global Change Monitoring

Mass Balance Computation in the Space Domain Using GRACE Data

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Gravimetric mass balance (GMB) is important for climatological applications. E.g. Recent contributions of glaciers and ice caps to sea level rise

High mountain Asia mass balance determination

T Jacob et al. 2012
GMB of the Antarctic Ice Sheet on April 2002, unit: kg/m², area: 2217-2642 km² relative to a modelled reference value, defined to be the GRACE-derived mass as of 2009-01-01

GMB of the Antarctic Ice Sheet on July 2016, unit: kg/m², area: 2217-2642 km² relative to a modelled reference value, defined to be the GRACE-derived mass as of 2009-01-01

GMB Observation

- Gravimetry mass balance (GMB) cannot be directly observed from the space.
- Only the change of Stokes coefficients can be observed

https://www.nasa.gov/mission_pages/Grace/multimedia/pia04236.html#.WyVJKEQV9gM
GMB Computation

How to compute the mass balance (MB) using the Stokes coefficients?

MB within an area B is the integral of the density change:

$$\Delta m_B = \int_B \Delta \sigma \, dA$$

The density change at a certain point can be derived from the change of the Stokes coefficients, via spherical harmonics (Wahr and Molenaar 1998):

$$\Delta \sigma(\varphi, \lambda) = \frac{R \rho_E}{3} \sum_{l=0}^{\infty} \sum_{m=0}^{l} Y_{lm}(\varphi, \lambda) \Delta c_{lm}$$

- Even if $\Delta \sigma$ can be given point-wisely, the value would be meaningless
- Applying the spatial averaging (integration) for $\Delta \sigma$ within an target area is necessary
Examples of the global density change converted to the equivalent water height

- Noises, “stripes” can be seen

Global EWH change on December 2016, **non filter** applied (unit: cm)

Based on the monthly solution from CNES/GRGS, reference value: the mean between 2002 and 2013
Examples of the global density change converted to the equivalent water height

- Spatial resolution of MB recovered from GRACE is around 200 km
- MB cannot be a point measurement, but a spatial average (Swenson & John 2002)

Global EWH change on December 2016, Gaussian filter applied (unit: cm)

based on the monthly solution from CNES/GRGS, reference value: the mean between 2002 and 2013
GMB Computation

Traditional approach – a spectral/frequency-domain method

• Applying the spatial averaging kernel (Characteristic function of the area) (Swenson & John 2002)

\[
\chi_{lm} = \int_B \overline{Y}_{lm}(\varphi, \lambda) \, dA \quad \Delta m_B = \frac{R \rho_E}{3} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \chi_{lm} \Delta c_{lm}
\]

• The integration of the Legendre functions is involved

• Using recursive formulas, known as Paul’s recursion (Paul 1978), to compute \( \chi_{lm} \)

\[
\overline{I}_{lm}(t_1, t_2) = \int_{t_1}^{t_2} \overline{P}_{lm}(t) \, dt , \quad \text{with} \ t = \sin \varphi
\]

Disadvantages:

• Longer mantissa for the computation significantly lose efficiency

• Instability, especially in the latitude range of 40°N - 70°N and 40°S - 70°S
Instead of compute the integration with a lot of recursions, can we make a direct numerical evaluation of the integration?

• Newly proposed method: Space-domain approach using numerical quadrature

Comparison between these two approaches

• Frequency domain
• Space domain

For the computation of

• Integral of the Legendre functions (ILF): \( \int_{t_1}^{t_2} \overline{P}_{lm}(t) \, dt \)
• MB: \( \int_{B} \Delta \sigma \, dA \)

In term of

• Precision of the results
• Stability of the results
• Computational efficiency
In order to evaluate the precision and determine the stability, a reference data set is needed, it requires:

- Precision: same as the length of the “double-float” number
- No instabilities

However...

- For ILF, we do not have a complete reference data set elsewhere
- For MB, Stokes coefficients contribute error, different filtering and data source lead to different result

Luckily...

- We can use sufficiently long mantissa (≥108 significant digits for $l_{\text{max}} = 100$) to compute ILF in the spectral domain, the result is accurate
- Using this ILF, we can compute MB and treat it as the true value
How is MB computed in the spectral domain?

- The Stokes coefficients and the characteristic function are multiplied element-wisely.
- For patches with the same latitude range, the characteristic function only needs to be computed once.

Disadvantages:
- Computationally inefficient: each patch has different characteristic function.
- Instability.
Instability

Minimum precision of ILF computed with double float number integrated every 5°, presented in the number of significant digits

<table>
<thead>
<tr>
<th>Integral Range</th>
<th>0°</th>
<th>5°N</th>
<th>10°N</th>
<th>15°N</th>
<th>20°N</th>
<th>25°N</th>
<th>30°N</th>
<th>35°N</th>
<th>40°N</th>
<th>45°N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precision</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>11</td>
<td>10</td>
<td>11</td>
<td>8</td>
<td>7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Integral Range</td>
<td>45°N</td>
<td>50°N</td>
<td>55°N</td>
<td>60°N</td>
<td>65°N</td>
<td>70°N</td>
<td>75°N</td>
<td>80°N</td>
<td>85°N</td>
<td>90°N</td>
</tr>
<tr>
<td>Precision</td>
<td>0</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td>7</td>
<td>8</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

- Same condition in the Southern Hemisphere
- Optimized recursion formula applied near the Pole
- E: (wrong values occur), indicates low precision and instabilities within the latitude range from 40°N to 70°N, starts from the integral of the tesseral harmonics
**Instability**

Precision of some representative values of $\tilde{I}_{ll}$ (integrals of tesseral harmonics) presented in the number of significant digits

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\varphi_1$</th>
<th>40°N</th>
<th>45°N</th>
<th>50°N</th>
<th>55°N</th>
<th>60°N</th>
<th>65°N</th>
<th>70°N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varphi_2$</td>
<td>45°N</td>
<td>50°N</td>
<td>55°N</td>
<td>60°N</td>
<td>65°N</td>
<td>70°N</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>14</td>
<td>14</td>
<td>13</td>
<td>12</td>
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<td>13</td>
<td>12</td>
<td>12</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>12</td>
<td>11</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>11</td>
<td>9</td>
<td>8</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>11</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>E</td>
<td></td>
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</tr>
<tr>
<td>60</td>
<td>9</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>E</td>
<td>E</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>7</td>
<td>3</td>
<td>0</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>6</td>
<td>2</td>
<td>E</td>
<td>E</td>
<td>E</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>5</td>
<td>0</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Instability starts from the **Middle latitude**
- Due to the property of recursion – later results depend on the previous results, errors from the integral of tesseral harmonics bring the instabilities to the final result
Minimum mantissa required for the recursive algorithm for the computation of ILF up to 100 degree, presented in the number of significant digits.

<table>
<thead>
<tr>
<th>( \varphi_1 )</th>
<th>0°</th>
<th>5°N</th>
<th>10°N</th>
<th>15°N</th>
<th>20°N</th>
<th>25°N</th>
<th>30°N</th>
<th>35°N</th>
<th>40°N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_2 )</td>
<td>5°N</td>
<td>10°N</td>
<td>15°N</td>
<td>20°N</td>
<td>25°N</td>
<td>30°N</td>
<td>35°N</td>
<td>40°N</td>
<td>45°N</td>
</tr>
<tr>
<td>Mantissa</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \varphi_1 )</th>
<th>45°N</th>
<th>50°N</th>
<th>55°N</th>
<th>60°N</th>
<th>65°N</th>
<th>70°N</th>
<th>75°N</th>
<th>80°N</th>
<th>85°N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_2 )</td>
<td>50°N</td>
<td>55°N</td>
<td>60°N</td>
<td>65°N</td>
<td>70°N</td>
<td>75°N</td>
<td>80°N</td>
<td>85°N</td>
<td>90°N</td>
</tr>
<tr>
<td>Mantissa</td>
<td>15</td>
<td>19</td>
<td>24</td>
<td>30</td>
<td>38</td>
<td>47</td>
<td>60</td>
<td>77</td>
<td>108</td>
</tr>
</tbody>
</table>

- Shorter mantissa required at lower latitudes
- The number differs when:
  - \( l_{\text{max}} \) changes
  - The integral range changes

When the mantissa is sufficiently long, would the results be perfect?
### Spectral Domain

**Latitude range:** 0° - 5°N, **Longitude range:** 2.5°W - 2.5°E

- Ideally:
  \[ \chi_B(\xi) = \begin{cases} 
  1 & \text{for } \xi \in B \\
  0 & \text{for } \xi \notin B 
\end{cases} \]

- The equiangular patch is in the "blue ring"

- Inside the patch, is almost 0

- Outside the patch there are negative values forming the disturbance (waves)

- Due to truncation

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \chi_{lm} \Delta c_{lm} \sum_{l=0}^{100} \cdots
\]
• More waves approaching the Pole, indicating the truncation has stronger influence

• The exact integration of an approximate characteristic function does not perform perfectly

latitude range: $10^\circ$S - $15^\circ$S, longitude range: $150^\circ$W - $145^\circ$W
Numerical Quadrature

Newton-Cotes Quadrature

\[ F(x_1, x_2, y_1, y_2) \triangleq \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \, dx \, dy \approx k_x \, k_y \, h_x \, h_y \sum_{j=0}^{J_x} v_{x_j} \, f_{x_j} \cdot \sum_{j=0}^{J_y} v_{y_j} \, f_{y_j} \]

Visualization of the weights of Bool’s rule distributed on an equiangular patch

- The location of the weights is evenly distributed, “linear”
- The value of the weights increases from the edge to the center
- For ILF computation, it estimates the integration of the exact characteristic function

The size of the red dots (or the brightness of the colorful patches) indicates:

- where the evaluation takes place
- how high the weight is

Rafelski, (1984)
Numerical Quadrature

Gaussian Quadrature

\[
G(x_1, x_2, y_1, y_2) \doteq \int_{-1}^{1} \int_{-1}^{1} g(x, y) \, dx \, dy \approx k_x k_y \sum_{j=0}^{J_x} w_{x_j} g_{x_j} \cdot \sum_{j=0}^{J_y} w_{y_j} g_{y_j}
\]

Visualization of the weights distributed on an equiangular patch

- The location of the weights is not evenly distributed, “nonlinear”
- The value of the weights also increases from the edge to the center
- But the distribution gets denser towards the edge
- Requires fewer evaluations than commonly used Newton-Cotes quadratures
Comparison

Minimum possible number of points required for two numerical quadrature rules ILF computation, for different latitude range with the size of 5°

<table>
<thead>
<tr>
<th>Latitude Range</th>
<th>Bool’s Rule</th>
<th>Gaussian Quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>0° - 20°N</td>
<td>57</td>
<td>8</td>
</tr>
<tr>
<td>20°N - 30°N</td>
<td>57</td>
<td>9</td>
</tr>
<tr>
<td>30°N - 40°N</td>
<td>61</td>
<td>9</td>
</tr>
<tr>
<td>40°N - 45°N</td>
<td>69</td>
<td>9</td>
</tr>
<tr>
<td>45°N - 70°N</td>
<td>85</td>
<td>9</td>
</tr>
<tr>
<td>70°N - 75°N</td>
<td>85</td>
<td>10</td>
</tr>
<tr>
<td>75°N - 80°N</td>
<td>117</td>
<td>12</td>
</tr>
<tr>
<td>80°N - 85°N</td>
<td>169</td>
<td>16</td>
</tr>
<tr>
<td>85°N - 90°N</td>
<td>341</td>
<td>20</td>
</tr>
</tbody>
</table>

- In general, Gaussian quadrature requires less than half as many evaluations as commonly used Newton-Cotes quadrature does
Comparison

Efficiency comparison of two numerical quadrature rules for ILF computation from 90°S to 90°N

<table>
<thead>
<tr>
<th>Name of the rule</th>
<th>Number of Points (nodes)</th>
<th>Averaged Computational Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boole’s Rule</td>
<td>341</td>
<td>$6.1 \times 10^{-2}$ s/IR</td>
</tr>
<tr>
<td>Gaussian Quadrature</td>
<td>20</td>
<td>$4.3 \times 10^{-3}$ s/IR</td>
</tr>
</tbody>
</table>

IR: Integral Range (with the size of 5° along the latitude)

- For space-domain approach, Gaussian quadrature has better performance
- Gaussian quadrature computes *hundreds* times as fast as Bool’s rule does
Comparison

Efficiency for ILF computation among listed approaches

<table>
<thead>
<tr>
<th>Approach</th>
<th>Spectral Domain Sufficient Mantissa</th>
<th>Space Domain Boole’s Rule</th>
<th>Space Domain Gaussian quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Averaged Computational Time</td>
<td>$1.7 \times 10^2$ s/IR</td>
<td>$6.1 \times 10^{-2}$ s/IR</td>
<td>$4.3 \times 10^{-3}$ s/IR</td>
</tr>
</tbody>
</table>

Sufficient long Mantissa: 89 significant digits, for $l_{\text{max}} = 80$

Efficiency for the global GMB computation between the spectral-domain approach and the space-domain approach

<table>
<thead>
<tr>
<th>Approach</th>
<th>Spectral Domain Sufficient Mantissa</th>
<th>Space Domain Gaussian quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Averaged Computational Time</td>
<td>$4.2 \times 10^1$ s/B</td>
<td>$5.2 \times 10^{-2}$ s/B</td>
</tr>
</tbody>
</table>

B: An equiangular patch on the Earth’s surface with a size of $5^\circ \times 5^\circ$
Global GMB on December 2016, unit: Gt
computed in the spectral domain with sufficiently long mantissa (89 significant digits)
Global GMB on December 2016, unit: Gt computed in the space domain using Gaussian quadrature (20 points)
Average precision of the global GMB from April 2002 to July 2016 computed in the space domain using Gaussian quadrature (20 points) presented in the number of significant digits.
The goal is to reach 6 significant digits
100% of the computed GMB reaches the goal, 46% (almost) identical to the reference
Worst case (less than 1%): losing 4 to 5 significant digits
Conclusion

Precision:
• Spectral domain: precise only using sufficiently long mantissa
• Space domain: as precise as the spectral-domain approach, by choosing an appropriate number of evaluation points

Stability:
• Spectral domain: Instability happens when the mantissa is not sufficiently long
• Space domain: Stable

Efficiency:
• Spectral domain: Not efficient, incredibly slow
• Space domain: Very efficient; Gaussian quadrature is almost 1000 times faster than the spectral-domain approach, and around 100 times faster than commonly used Newton-Cotes quadrature

Others:
• The evaluation of the integration along the longitude direction can be optimized by analytic solution
Outlook

• Apply advanced numerical analysis, e.g. Romberg's method
• The algorithm can be further optimized: at different latitude, the number of the evaluation point for the numerical quadrature differs

2. Jacob, T. & et al. (2012), Nature 000, 1-5, URL: http://dx.doi.org/10.1038/nature10847


4. CNES/GRGS (2016), Monthly solution of the Stokes coefficients, URL: https://grace.obs-mip.fr/


Blood Moon

Thank you for the attention!